THE CALCULUS OF VARIATIONS FOR MULTIPLE INTEGRALS*

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Introduction. On page 158 of his article on the calculus of variations in the Encyclopédie des Sciences Mathématiques Lecat [1916]† makes a statement concerning the theory for multiple integrals which may be translated as follows:

"The theory of the calculus of variations for multiple integrals is much less advanced than for simple integrals. Some of the theory for simple integrals is easily carried over, it is true, but for the most part difficulties increase, largely as a result of the fact that ordinary differential equations must be replaced by partial differential equations. One cannot develop a satisfactory theory of the problems of Lagrange or Mayer for multiple integrals until one has established numerous properties of partial differential equations which are now lacking. Multiple integrals have been made the objects of very few rigorous researches."

These statements are still to a large extent applicable in 1941. Though progress has been made in the twenty-five years which have elapsed since 1916, the theory of the calculus of variations for multiple integrals is still far from complete relative to that for simple integrals. The purpose of this paper is to describe some of the progress which has been made, and to call attention to some of the hiati in the theory which still remain. The field is a large one and the author, in confining himself to those portions of the theory which have interested him especially, must omit much of importance which has been of great interest to others.

1. Formulation of the problem. The integral to be studied is

$$I = \int_{\mathfrak{X}} f(x, y, y')d\mathfrak{X} \tag{1}$$

with the understanding that

$$x = (x_1, \cdots, x_m),\ y = (y_1, \cdots, y_n),\ y' = \|\partial y_i/\partial x_\alpha\| = \|y_{i\alpha}\|,$$
$$(\alpha = 1, \cdots, m;\ i = 1, \cdots, n),$$
$$d\mathfrak{X} = dx_1 \cdots dx_m.$$

The manifolds on which the values of the integral are to be taken are defined

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† The references in the text are to the chronological bibliography at the end of this paper. The references there are taken from a "Bibliography for the theory of multiple integrals in the calculus of variations" which is scheduled to appear about the time this paper will be printed. This Bibliography is the last section of a book entitled "Contributions to the calculus of variations 1938–1941" edited by members of the Department of Mathematics of the University of Chicago. It contains about 350 titles and is divided into three parts in which the papers are listed chronologically, alphabetically, and by subject matter, respectively.

by functions of the form

$$y(x), \qquad (x \text{ on } \mathfrak{X}), \tag{2}$$

where $\mathfrak{X}$ is the range in x-space on which the set of functions $y(x)$ is defined. The boundary of $\mathfrak{X}$ will be denoted by $\mathfrak{B}$. The problem is then to find in a class of m-dimensional spaces (2) defined over the same range $\mathfrak{X}$ and having identical values on the boundary $\mathfrak{B}$ of $\mathfrak{X}$, one which minimizes the integral I.

The first variation of the integral I on a space (2) is

$$I_1 = \int_{\mathfrak{X}} (f_i \eta_i + f_{i\alpha} \eta_{i\alpha})\, d\mathfrak{X}, \tag{3}$$

with the notations

$$f_i = \partial f/\partial y_i, \quad f_{i\alpha} = \partial f/\partial y_{i\alpha}, \quad \eta_{i\alpha} = \partial \eta_i/\partial x_\alpha.$$

It is found as usual by replacing $y(x)$ by $y(x)+a\eta(x)$ in I, differentiating with respect to a and setting $a=0$. In formula (3) the repeated indices i and α indicate, as customary in tensor analysis, that sums for $i=1, \cdots, n$ and $\alpha=1, \cdots, m$ are to be taken. The symbol $\eta(x)$ stands for a set of "admissible" functions $\eta_i(x)$ defined on the range $\mathfrak{X}$ and vanishing on the boundary $\mathfrak{B}$ of $\mathfrak{X}$. Similarly the second variation, found by differentiating twice with respect to a and setting $a=0$, has the form

$$I_2 = \int_{\mathfrak{X}} 2\omega(x, \eta, \eta')\, d\mathfrak{X} \tag{4}$$

in which $\eta' = \|\eta_{i\alpha}\|$ and 2ω is the homogeneous quadratic form in the elements of η, η' whose coefficients are the usual second derivatives of the function f with respect to the elements of y and y'. It is evident that on a minimizing space (2) the conditions $I_1=0$, $I_2 \geqq 0$ must be satisfied for all choices of admissible sets η which vanish on the boundary $\mathfrak{B}$ of $\mathfrak{X}$.

It should be noticed that the very general problem formulated in the preceding paragraphs includes, when $(m, n)=(1, n)$, the classical problem of the calculus of variations with fixed end points for simple integrals, and when $(m, n)=(m, 1)$ the simplest problem with fixed boundaries for multiple integrals.

It is understood in the following pages that without further specification the integrand function f is supposed to have continuous partial derivatives in a region $\mathfrak{R}$ of values (x, y, y') of as many orders as may be needed at any time to carry through the part of the theory which is being studied. A set (x, y, y') interior to $\mathfrak{R}$ is called "admissible." All of the sets (x, y, y') beonging to the spaces (2) under consideration are supposed to be admissible.

2. Conditions for a minimum associated with the first variation. The Lagrange equations for the integral I are the equations

$$df_{i\alpha}/dx_\alpha = f_i, \qquad (i = 1, \cdots, n). \tag{5}$$

A space (2) having continuous second derivatives and satisfying these equations is called an extremal. It has long been known that each piece of a minimizing space (2) on which the functions $y(x)$ have continuous second derivatives must be an extremal. The proof for the problem of Plateau, for which $(m, n) = (2, 1)$ and $f = [1+y_{11}^2+y_{12}^2]^{1/2}$, was made by Lagrange [1762], and for the general case $(m, n) = (2, 1)$ also by Lagrange [1806]. The proofs for more general values m, n are quite similar and have been made by many writers. The second derivatives of the space (2) appear both in the transformations of the first variation used in the proof, and in the resulting equations (5) themselves.

For a piece of a minimizing space only assumed to have continuous first derivatives Haar [1919, 1930] has deduced independently some interesting equations. His results are also a consequence of a paper by Mason [1905]. The simplest form of these results for application to the general integral I was suggested by Coral [1937]. Let $\mathfrak{X}_0$ be a sub-region of the region $\mathfrak{X}$ with boundary $\mathfrak{B}_0$. Let the direction cosines of the outer normal to $\mathfrak{B}_0$ be designated by l_α, and let $d\mathfrak{B}$ be the generalized element of area on $\mathfrak{B}_0$. Then on every region $\mathfrak{X}_0$ on which a minimizing space (2) has continuous first derivatives the equations

$$\int_{\mathfrak{X}_0} f_i d\mathfrak{X} = \int_{\mathfrak{B}_0} f_{i\alpha} l_\alpha d\mathfrak{B}, \qquad (i = 1, \cdots, n), \tag{6}$$

must hold. The equations of Lagrange are consequences of these when the minimizing space has continuous second derivatives.

Haar deduced his results especially for the cases when $m=2$ and $m=3$. For the general integral I the Haar-Coral equations (6), and a number of other interesting results, are consequences of methods introduced by Hestenes [1941] and pursued by Carson [1941]. The latter considers the equation

$$\int_{\mathfrak{X}} (w\eta + v_\alpha \partial\eta/\partial x_\alpha) d\mathfrak{X} = \int_{\mathfrak{B}} v_\alpha l_\alpha \eta d\mathfrak{B} \tag{7}$$

in which w, v_α are given functions of the variables x, and the other symbols have the same significance as before. Following the procedure of Hestenes for the two-dimensional case Carson finds a variety of necessary and sufficient conditions that the equation (7) shall hold for all admissible sets of functions η. From these he shows that the Haar-Coral equations are necessary and sufficient in order that the first variation (3) shall have the value zero for all admissible sets η vanishing on the boundary $\mathfrak{B}$, and he finds also an independent proof of the analogue of Green's theorem for higher spaces. His results are applicable further to the deduction of the ridge condition mentioned in the following paragraph.

A ridge on a continuous space (2) is an $(m-1)$ dimensional boundary between two pieces of the space on each of which the space has continuous first derivatives. On the ridge itself these derivatives are not continuous. The so-called ridge condition says that across a ridge on a minimizing space (2) the sums $f_{i\alpha}l_\alpha$ must be continuous, where the arguments of the derivatives $f_{i\alpha}$ are

those belonging to the minimizing space and the l_α are the direction cosines of the normal to the ridge. The ridge conditions seem to have been deduced first for multiple integrals by Kobb [1892, 1893] for the parametric case in 3-space. A more recent proof for non-parametric problems when $(m, n)=(2, 1)$, with integrands of the form $f(x, y, z, p, q)$, has been given by Bliss [1939]. By transforming this latter problem into parametric form and applying the conditions of Kobb it is found that across a ridge on a minimizing surface $z(x, y)$ the three determinants of the matrix

$$\left\| \begin{matrix} f_p & f_q & pf_p + qf_q - f \\ dx & dy & dz \end{matrix} \right\|$$

must be continuous when dx, dy, dz define the tangent to the ridge curve and the arguments of f are $x, y, z, p=z_x, q=z_y$. The general case for arbitrary values (m, n) has been treated by Powell [1931]. For the problems of Dirichlet and Plateau the ridge conditions imply that a minimizing surface can have no ridges. The conditions have an important application in the proof of the analogue of the condition of Jacobi by the method suggested by Bliss (1939).

3. Conditions analogous to those of Weierstrass and Legendre. The function $E(x, y, y', Y')$ of Weierstrass is by definition

$$E = f(x, y, Y') - f(x, y, y') - (Y_{i\alpha} - y_{i\alpha})f_{i\alpha}(x, y, y').$$

It is provable that at every element (x, y, y') of a minimizing space (2) the condition $E \geqq 0$ must be satisfied for all sets Y' such that (x, y, Y') is admissible and the matrix $\| Y_{i\alpha} - y_{i\alpha} \|$ has rank 1. The proof for multiple integrals of this so-called condition of Weierstrass was first made for the case $(m, n)=(2, 1)$ by Levi [1915], and for the general (m, n) case by McShane [1931]. Simpler proofs have been given for $(m, n)=(2, n)$ by Coral [1937] and for arbitrary (m, n) by Graves [1939].

The requirement that the matrix $\| Y_{i\alpha} - y_{i\alpha} \|$ shall be of rank 1 is no restriction when either of the integers m or n is unity, since the matrix has then only one row or one column. The requirement is a restriction, however, when $m \geqq 2$ and $n \geqq 2$, and at first it may seem an unnatural one. But the following simple example for the case $(m, n)=(2, 2)$, proposed by Reid, shows that from the nature of the problem the restriction is unavoidable. The integral I whose integrand is $f=y_{11}y_{22}-y_{12}y_{21}$ is independent of the path, since its Lagrange equations are identically satisfied. It is therefore minimized by every space (2) with suitable continuity properties. Its E-function, which has the form

$$E = (Y_{11} - y_{11})(Y_{22} - y_{22}) - (Y_{12} - y_{12})(Y_{21} - y_{21}),$$

vanishes when the matrix $\| Y_{i\alpha} - y_{i\alpha} \|$ is of rank 1, and when y' is given E may be made either positive or negative by properly choosing Y'. Thus it is hopeless to expect to prove the necessity of the condition that on a minimizing space (2) we must have $E \geqq 0$ for unrestricted values Y'.

The necessary condition of Legendre states that at every set (x, y, y') belonging to a minimizing space (2) the quadratic form $f_{i\alpha,k\beta}\pi_{i\alpha}\pi_{k\beta}$ must be non-negative for all sets of values $\pi_{i\alpha}$ such that the matrix $\|\pi_{i\alpha}\|$ has rank 1. This condition is a consequence of that of Weierstrass by a well-known method which uses Taylor's formula (see Bliss [1939]). It was proved for the case $(m, n) = (2, 1)$ by Mason [1907]. If the quadratic form is everywhere positive for all of the sets $\pi \neq 0$ specified, the space (2) is said to satisfy the "strengthened" condition of Legendre. From a rather complicated transformation of the second variation by Clebsch [1859] Hadamard [1902, 1905] inferred that the necessity of the non-negativeness of the quadratic form of Legendre could be proved only for matrices $\|\pi_{i\alpha}\|$ of rank 1. It was this remark which called attention to the similar restriction in the case of the condition of Weierstrass.

4. Conditions analogous to that of Jacobi. For the problem formulated in section 1 above the so-called accessory equations belonging to a space (2) are the Lagrange equations formed from the integrand 2ω of the second variation,

$$J_i(\eta) = d\omega_{i\alpha}/dx_\alpha - \omega_i = 0 \tag{8}$$

with the notations

$$\omega_i = \partial\omega/\partial\eta_i, \quad \omega_{i\alpha} = \partial\omega/\partial\eta_{i\alpha}.$$

They are linear and homogeneous in the functions η and the first and second derivatives of these functions with respect to the variables x. The coefficients are expressible in terms of derivatives of the integrand f which are well-defined functions of the variables x when the functions $y(x)$ occurring in them and belonging to the space (2) have continuous second derivatives. The necessary condition of Jacobi states that on a minimizing extremal space (2) there can be no solution $\eta(x)$ of the accessory equations (8) defined over a sub-region $\mathfrak{X}_0$ of $\mathfrak{X}$, vanishing on the boundary $\mathfrak{B}_0$ of $\mathfrak{X}_0$, and having derivatives $\eta_{i\alpha} = \partial\eta_i/\partial x_\alpha$ not all zero at a point of $\mathfrak{B}_0$ interior to $\mathfrak{X}$.

A relatively simple proof of this theorem for the case $(m, n) = (2, 1)$ was given by Sommerfeld [1899] following a method suggested by Schwarz [1885]. Perhaps the simplest modern proof is analogous to one suggested for simpler cases by Bliss. Assume that there exists a solution of the accessory equations with the properties described in the condition. Extend the definition of $\eta(x)$ to be $\eta \equiv 0$ over the part of $\mathfrak{X}$ outside of $\mathfrak{X}_0$. For this extended $\eta(x)$ it can be proved that the second variation has the value zero, and that the set $\eta(x)$ does not satisfy along $\mathfrak{B}_0$ the ridge condition for a minimum of the second variation. Thus zero is not the smallest value possible for the second variation on the space (2), and this space cannot minimize I since on a minimizing space the second variation is necessarily positive. Proofs have been made by this method for the case $(m, n) = (2, 1)$ by Kubota [1916], and for arbitrary (m, n) by Raab [1932]. Reid [1939] proves the condition with more general regions $\mathfrak{X}_0$ and boundaries $\mathfrak{B}_0$. Haar [1930] replaces the accessory equations (8) by equations analogous to those deduced by him from the first variation.

The early literature of the calculus of variations concerning the conditions of Legendre and Jacobi, in forms not always clearly stated, is voluminous and complicated. Its purpose is for the most part to attain necessary conditions for a minimum by transformations of the second variation or other methods similar to those used by Legendre and Jacobi themselves in simpler cases. For this literature the reader is referred to the sections of the Bibliography mentioned above on the Jacobi condition and the second variation.

Boundary value problems associated with the second variation have been formulated in a number of related forms. One of them is the problem of finding a solution η, λ of the equations and boundary conditions

$$J_i(\eta) + \lambda\eta_i = 0, \quad \eta \equiv 0 \text{ on } \mathfrak{B}, \tag{9}$$

where λ is a parameter. A second necessary condition for a minimum, related to that of Jacobi, states that on a minimizing space (2) satisfying the Lagrange equations and the strengthened condition of Legendre there can be no solution of the boundary value problem (9) with a negative value of the parameter λ. The first proof of a theorem like this seems to have been given by Schwarz (1885) for the Plateau problem. The case $(m, n) = (2, 1)$, with a different boundary value problem, has been discussed by Lichtenstein [1915, 1917, 1920] and Picone [1922]. Reid [1932] relates the different points of view hitherto adopted, but further possibilities might also well be studied. When the minimum problem of section 1 is replaced by one with variable boundaries a necessary condition similar to that of the first paragraph of this section is in general lacking. Conditions arising from boundary value problems of the second variation have, however, been considered for the case $(m, n) = (2, 1)$ by Lichtenstein [1919] and Simmons [1926], and for the case $(m, n) = (m, 1)$ by Simmons [1934].

5. Fields and sufficiency proofs. A field for the integral I of section 1 is defined to be a region F of xy-space with a set of slope functions $p_{i\alpha}(x, y)$ and an invariant integral

$$I^* = \int D(x, y, y')d\mathfrak{X}$$

such that

$$D(x, y, p) = f(x, y, p), \quad D_{i\alpha}(x, y, p) = f_{i\alpha}(x, y, p). \tag{10}$$

An integral is said to be invariant if it has the same value on all spaces (2) having a common boundary. Since every space (2) is a minimizing space for an invariant integral, and therefore satisfies the Lagrange equations of the integral, it is provable with the help of equations (10) and a very simple argument that every solution (2) of the equations

$$\partial y_i/\partial x_\alpha = p_{i\alpha}(x, y) \tag{11}$$

is a solution also of the equations of Lagrange for the integral I. A space (2)

which satisfies equations (11) is called an *extremal of the field*, and the equations (11) are called the differential equations of the field. It follows readily from the first equation (10) that on an extremal space $\mathfrak{E}$ of the field $I(\mathfrak{E}) = I^*(\mathfrak{E})$.

The importance and use of fields in making sufficiency proofs was first emphasized by Weierstrass for simpler cases. In general let $\mathfrak{E}$ be an extremal space imbedded in a field $\mathfrak{F}$, and let $\mathfrak{S}$ be an arbitrary space of the form (2) in the field having the same boundary as $\mathfrak{E}$. Then from the properties of the integral I^* it follows readily that

$$(12) \qquad I(\mathfrak{S}) - I(\mathfrak{E}) = I(\mathfrak{S}) - I^*(\mathfrak{E}) = I(\mathfrak{S}) - I^*(\mathfrak{S}) = \int_{\mathfrak{S}} (f - D)d\mathfrak{X}.$$

It is evident from the last equation that if the function $f - D$ is everywhere non-negative then $I(\mathfrak{E})$ is a minimum. If a space $\mathfrak{E}$ whose minimizing properties are to be established is given in advance one may seek to imbed $\mathfrak{E}$ in a field $\mathfrak{F}$ as an extremal of the field, and then to show that the integrand $f - D$ is non-negative. For the case of simple integrals, when $(m, n) = (1, n)$, it is well known that suitably strengthened conditions of Lagrange, Weierstrass, Legendre, and Jacobi insure the possibility of carrying through this process. For the case $(m, n) = (2, 1)$ some of the corresponding theorems have been established by Lichtenstein [1917] and other writers, but in general the theory for multiple integrals is still only fragmentary.

The differential equations (11) of a field $\mathfrak{F}$ may be incompatible and have no solution, in which case the field seems to have little interest, or they may have a particular space $\mathfrak{E}$ as a solution but not be compatible everywhere in $\mathfrak{F}$, or they may be "completely integrable" in $\mathfrak{F}$ and define an n-parameter family of extremals

$$(13) \qquad y_i = y_i(x_1, \cdots, x_m, c_1, \cdots, c_n) = y_i(x, c)$$

simply covering the region $\mathfrak{F}$. A well-known necessary and sufficient condition for this last to happen is that the integrability conditions

$$\partial p_{i\alpha}/\partial x_\beta + p_{k\beta}\partial p_{i\alpha}/\partial y_k = \partial p_{i\beta}/\partial x_\alpha + p_{k\alpha}\partial p_{i\beta}/\partial y_k$$

are satisfied identically in $\mathfrak{F}$. The field is then called a *field of extremals*. In that case the equations (13) have solutions $c_i(x, y)$ and the slope functions of the field are the slope functions

$$p_{i\alpha}(x, y) = y_{i\alpha}[x, c(x, y)]$$

of the family.

6. The construction of fields. One of the first problems which arises in the construction of fields is the determination of the possible forms for the invariant integral I^*. Two ingenious special types of such integrals and the fields associated with them have been studied by Carathéodory [1922, 1929] and his associates, and by Weyl [1935]. The integrand D for the invariant integral of

Carathéodory is a functional determinant, and for that of Weyl is a divergence. The most general integrand D of an invariant integral was determined by Smiley in 1938 as a sum of determinants of all orders of the matrix $\|y_{i\alpha}\|$ with coefficients functions of the variables x, y satisfying a certain system of partial differential equations. The integrand D is evidently a polynomial in the variables $y' = \|y_{i\alpha}\|$ of degree equal to the smaller ν of the integers m and n. The conditions of Smiley are consequences of the fact that a necessary and sufficient condition for an integral I^* to be invariant is that the Lagrange equations formed with the integrand D of I^* shall be satisfied identically in the variables x, y, y', y''.

Every integrand function D belonging to a field must be related to the slope functions $p(x, y)$ of the field by the equations (10). If D is expanded in powers of the variables $y' - p$ by Taylor's formula the terms of degree less than 2 are completely determined by the conditions (10), and D has the form

$$D = f(x, y, p) + (y_{i\alpha} - p_{i\alpha}) f_{i\alpha}(x, y, p) + D_2 + \cdots, \tag{14}$$

where D_k $(k \geqq 2)$ is a symbol for the terms of order k in the expansion of D and is a sum of determinants of order k of the matrix $y' - p$ with coefficients which are functions of the variables x, y. The functions $p(x, y)$ and these coefficients must satisfy the equations of Smiley deduced for this form of D. The difference $f - D$ has the form

$$f - D = E(x, y, p, y') - D_2 - \cdots \cdot \tag{15}$$

If the E-function is also expanded in powers of the variables $y' - p$ it is found that

$$f - D = (y_{i\alpha} - p_{i\alpha})(y_{k\beta} - p_{k\beta})(f_{i\alpha, k\beta} - D_{i\alpha, k\beta}) + (f_3 - D_3) + \cdots \tag{16}$$

where the arguments of the second derivatives of f and D are x, y, p, and $f_k (k \geqq 2)$ is a symbol for the terms of order k in the expansion of f.

For simple integrals when $(m, n) = (1, n)$, and for multiple integrals with a single dependent variable when $(m, n) = (m, 1)$, the value of ν is unity, D is uniquely determined as the sum of the terms of orders < 2 in (14), and the sign of $f - D = E$ in (15) is that of E. For the case of simple integrals it can be proved from Smiley's equations that every field is a field of extremals. Except when $n = 1$ not every n-parameter family of extremals simply covering a region $\mathfrak{F}$ of xy-space has slope functions belonging to a field. The families belonging to fields are called Mayer families and are special in character, as is well known.

When $(m, n) = (m, 1)$ the integrals involved are multiple integrals whose integrands depend upon only one function y. In this case there may be fields which are not fields of extremals, but every one-parameter family of extremals simply covering a region $\mathfrak{F}$ of xy-space has slope functions forming a field over $\mathfrak{F}$.

The case $(m, n) = (2, 2)$ is the one which inspired the remarks of Hadamard mentioned above concerning the form of the conditions of Legendre. It was

studied in some detail by Alaoglu in 1938. The expansion (14) for D has in this case no terms of order greater than 2, and D_2 has the simple form

$$D_2 = B(x, y)[(y_{11} - p_{11})(y_{22} - p_{22}) - (y_{12} - p_{12})(y_{21} - p_{21})].$$

The slope functions p and the coefficient B must satisfy four partial differential equations of the first order. These equations doubtless have solutions not belonging to any field of extremals. But by a pleasantly skillful manipulation Alaoglu shows that when the functions $p(x, y)$ are given as the slope functions of a 2-parameter family of extremals two of the equations are identically satisfied and the remaining two are linear in B and its first derivatives and compatible. Thus the slope functions $p(x, y)$ of every two-parameter family of extremals simply covering a region $\mathfrak{F}$ in xy-space are the slope functions of an infinity of fields associated with the family.

The analysis of the invariant integrals associated with fields of extremals becomes much simpler if the coordinates x, y are replaced by x, c, where the c's are the parameters of the extremal family, and if the integrand f is replaced by $f-D$ where D is the integrand of some particular invariant integral. This last modification does not affect the solutions of the problem since the integral I^* is invariant. When these changes have been made and the new coordinates have been designated again by x, y the family of extremals of the field is the family $y_i = c_i$, the slope functions $p(x, y)$ of the family are all identically zero, and it turns out that

$$f(x, y, 0) \equiv 0, \quad f_{i\alpha}(x, y, 0) \equiv 0.$$

The equations (15) and (16) for the case $(m, n) = (2, 2)$ take the forms

$$(17) \quad f - D = f(x, y, y') + B(y_{11}y_{22} - y_{12}y_{21})$$
$$= y_{i\alpha}y_{k\beta}f_{i\alpha, k\beta}(x, y, 0) + B(y_{11}y_{22} - y_{12}y_{21}) + f_3 + \cdots.$$

The equations of Smiley are satisfied if and only if B is a function of the variables y alone. They will certainly be satisfied if B is a constant.

In order to find a field in which $f-D$ is non-negative we seek to determine the coefficient $B(y)$ so that the terms of the second order in the second expression (17) for $f-D$ form a positive definite quadratic form. So far this has not been done for the whole of the region $\mathfrak{F}$. But it can be shown that when the strengthened condition of Legendre is satisfied by an extremal space $\mathfrak{E}$ of a field then there exists for each particular point $(x, y)_0$ of $\mathfrak{E}$ a constant B such that at $(x, y)_0$ the quadratic terms in (17) are positive definite. The difference $f-D$ will then be non-negative at all points (x, y) of the field $\mathfrak{F}$ sufficiently near to $(x, y)_0$ and for all sufficiently small values of the variables y'. From the formula (12) it follows that each piece $\mathfrak{E}'$ of $\mathfrak{E}$ sufficiently near to $(x, y)_0$ will minimize the integral I relative to other spaces having sufficiently small values of the variables y' and passing through the boundary of $\mathfrak{E}'$. This means, in other words, that each sufficiently small piece $\mathfrak{E}'$ of an extremal $\mathfrak{E}$ of a field gives to the integral I a weak relative minimum.

The existence of a constant B which will made the second order terms in the expression (17) positive definite at a particular point $(x, y)_0$ is a consequence of a theorem on quadratic forms which has interested a number of writers.* This theorem says that if a quadratic form $P(z)$ in variables $z_1, \cdots, z_n$ with real coefficients is positive at every set of real values z not all zero making a second quadratic form $Q(z)$ vanish, then there exists a constant B such that the quadratic form $P(z)+BQ(z)$ is positive definite. The quadratic form in the second expression (17) with coefficients derivatives of f is positive for all sets $y_{i\alpha}$ not all zero making the determinant $y_{11}y_{22}-y_{12}y_{21}$ vanish, on account of the strengthened condition of Legendre which is supposed to hold on the extremal space $\mathfrak{E}$ under consideration. Thus by a suitable selection of a constant value for B the quadratic terms in the second expression (17) can be made positive definite at a particular point $(x, y)_0$, and $f-D$ will be positive for all sets (x, y, y') with (x, y) sufficiently near to $(x, y)_0$ and elements y' sufficiently small.

The theorem on quadratic forms mentioned in the last paragraph has been generalized by Hestenes and McShane [1940] so that similar arguments are effective for the case $(m, n)=(2, n)$, and one can perhaps treat the case $(m, n)=(m, 2)$ in a similar manner. An example has been given by Terpstra [1938] to show that when $m \geqq 3$, $n \geqq 3$ the generalization of the quadratic form theorem which might be expected to hold in general is no longer true. For these cases therefore the arguments made above would need to be modified in some way not yet discovered.

7. Conclusion. From what has been said above it is clear that the theory of the calculus of variations for multiple integrals is still very imperfect. For the cases $(m, n)=(1, n)$ involving simple integrals the equations of the extremals are ordinary differential equations for which existence and imbedding theorems are well known. If an extremal arc $\mathfrak{E}$ in the $(n+1)$ dimensional xy-space satisfies suitably strengthened conditions of Weierstrass, Legendre, and Jacobi, then $\mathfrak{E}$ belongs to an n-parameter family of extremal arcs whose slope functions form a field surrounding $\mathfrak{E}$ in which the difference $f-D$ is non-negative, and $I(\mathfrak{E})$ is therefore a minimum.

For the case $(m, n)=(2, 1)$ Lichtenstein [1917] proved a similar succession of theorems, using his boundary value problem form of Jacobi's condition. In this case therefore, and for this form of the condition of Jacobi, the theory is comparable to that for simple integrals.

Let us consider for the moment, in the general case (m, n), an extremal space $\mathfrak{E}$ satisfying the strengthened condition of Legendre and imbedded in an n-parameter family of extremals simply covering a region $\mathfrak{F}$ of xy-space. From the results of Smiley and Landers [1939] it follows that when $m \geqq 2$, $n \geqq 2$ there is an infinity of invariant integrals I^* which form fields in $\mathfrak{F}$ with the slope functions $p(x, y)$ of the extremal family. But only for special values (m, n) and

* Finsler [1937], Albert [1938], Reid [1938], Terpstra [1938], Hestenes and McShane [1940], Dines [1941].

for limited portions $\mathfrak{E}'$ of $\mathfrak{E}$ is it known that among the possible integrals I^* there are some such that the condition $f-D\geqq 0$ is satisfied in the field near $\mathfrak{E}'$, with the consequence that we can be sure that $I(\mathfrak{E}')$ is a minimum. This seems, however, a much too restricted result in any case. One would hope to be able to prove that the whole space $\mathfrak{E}$ has the minimizing property.

Weyl [1935] has given a proof that an extremal space $\mathfrak{E}$ with suitable properties is imbedded in a field of the type which he devised, and Boerner (1936) has a similar theorem for fields of the Carathéodory type. If the condition $f-D\geqq 0$ holds near $\mathfrak{E}$ in one of these fields then $I(\mathfrak{E})$ will be a minimum. But so far as I know this condition has not been proved to be necessary for a minimum in either case. The question as to whether $\mathfrak{E}$ can be imbedded in a field in which $f-D\geqq 0$ near $\mathfrak{E}$ is still unanswered.

There are many topics of interest and importance in connection with the theory of the calculus of variations for multiple integrals which are not discussed in the preceding pages. The reader is referred to the fourth section of the bibliography mentioned in the introduction above, in which the classification of papers is by subject matter. Among the topics there mentioned and recently studied, some of them in a large number of papers, are problems with variable edges, the inverse problem, the character of the solutions of the equations of Lagrange and Haar, existence theorems, the problem of Dirichlet, and especially the problem of Plateau.

Bibliography

1762 Lagrange, J. L., Essai d'une nouvelle méthode pour determiner les maxima et les minima des formules intégrales indéfinies, Miscellanea Taurinensia, vol. II, pp. 173–195; German translation in Ostwald's Klassiker der exakten Wissenschaften Nr. 47, pp. 20–24.

1806 Lagrange, J. L., Leçons sur le calcul des fonctions, 2nd ed., p. 471; Oeuvres X, p. 424.

1859 Clebsch, A., Über die zweite Variation vielfacher Integrale, Journal für die reine und angewandte Mathematik, vol. LVI, pp. 122–148.

1885 Schwarz, H. A., Über ein die Flächen kleinsten Flächeninhalts betreffendes Problem der Variationsrechnung. Festschrift zum Jubelgeburtstage des Herrn Karl Weierstrass. See also Schwarz, Gesammelte Mathematische Abhandlungen I, pp. 223–269.

1892 Kobb, G., Sur les maxima et les minima des intégrales doubles, Acta Matematica, vol. XVI, pp. 65–140.

1893 Kobb, G., Sur les maxima et les minima des intégrales doubles, Acta Matematica, vol. XVII, pp. 321–344.

1899 Sommerfeld, A., Bemerkungen zur Variationsrechnung, Jahresbericht der deutschen Mathematiker-Vereinigung, vol. VIII, pp. 188–193.

1902 Hadamard, J., Sur une question de calcul des variations, Bulletin de la Société mathématique de France, vol. XXX, pp. 253–256.

1905 Mason, M., Beweis eines Lemmas der Variationsrechnung, Mathematische Annalen, vol. XLI, pp. 450–452.

Hadamard, J., Sur quelques questions du calcul des variations, Bulletin de la Société Mathématique de France, vol. XXXIII, pp. 73–80.

1907 Mason, M., A necessary condition for an extremum of a multiple integral, Bulletin of the American Mathematical Society, vol. XIII, pp. 293–298.

1915 Levi, E. E., Sulla necessita della condizione di Weierstrass per l'estremo degli integrali doppi, Atti della Reale Accademia Nazionali dei Lincei, Rendiconti, vol. XXIV^2, pp. 353–359.

Lichtenstein, L., Die Jacobische Bedingung bei zwei-dimensionalen regulären Variationsproblemen, Sitzungsberichte der Berliner Mathematiker-Gesellschaft, vol. XIV, pp. 119–121.

1916 Kneser, A.-Lecat, M., Encyclopédie des Sciences mathématiques, Section II31, especially pp. 57–64.

Kubota, Die Jacobische Bedingung in der Variationsrechnung, Science Report, Tôhoku Imperial University, vol. V, pp. 241–247.

1917 Lichtenstein, L., Über zweidimensionale reguläre Variations-probleme, Monatschefte für Mathematik und Physik, vol. XXVIII, pp. 3–51.

1919 Haar, A., Über die Variation der Doppelintegrale, Journal für die reine und angewandte Mathematik, vol. CXLIX, pp. 1–18.

Lichtenstein, L., Zur Variationsrechnung, Nachrichten der königlichen Gesellschaft der Wissenschaften zu Göttingen, pp. 161–192.

1920 Lichtenstein, L., Untersuchungen über zwei-dimensionale reguläre Variationsprobleme, Mathematische Zeitschrift, vol. VI, pp. 26–51.

1922 Picone, M., Nuovo condizione necessaria per un estremo di un integrale doppio, Atti della Reale Accademia Nazionali dei Lincei, Rendiconti, vol. XXXI[1], pp. 46–48.

Carathéodory, C., Über ein Reziprozitätsgesetz der verallgemeinerten Legendreschen Transformation, Mathematische Annalen, vol. LXXXVI, pp. 272–275.

1926 Simmons, H. A., First and second variations of a double integral for the case of variable limits, Transactions of the American Mathematical Society, vol. XXVIII, pp. 235–251.

1929 Carathéodory, C., Über die Variationsrechnung bei mehrfachen Integralen, Acta Litterarum ac Scientiarum, Szeged, vol. IV, pp. 193–216.

1930 Haar, A., Zur Variationsrechnung, Abhandlungen aus dem mathematischen Seminar der Hamburgschen Universität, vol. VIII, pp. 1–27.

1931 Powell, J. E., Edge conditions for multiple integrals in the calculus of variations, Contributions to the Calculus of Variations 1931–1932, University of Chicago, pp. 1–58.

McShane, E. J., On the necessary conditions of Weierstrass in the multiple integral problem of the calculus of variations, Annals of Mathematics, vol. XXXII, pp. 578–590, 723–733.

1932 Raab, A. W., Jacobi's condition for multiple integral problems of the calculus of variations, Contributions to the Calculus of Variations 1931–1932, University of Chicago, pp. 445–471.

Reid, W. T., On boundary value problems associated with double integrals in the calculus of variations, American Journal of Mathematics, vol. LIV, pp. 791–801.

1934 Simmons, H. A., The first and second variations of an n-tuple integral in the case of variable limits, Transactions of the American Mathematical Society, vol. XXXVI, pp. 29–43.

1935 Weyl, H., Geodesic fields in the calculus of variations for multiple integrals, Annals of Mathematics, vol. XXXVI, pp. 607–629.

1936 Boerner, H., Über die Extremalen und geodätischen Felder in der Variationsrechnung der mehrfachen Integrale, Mathematische Annalen, vol. CXII, pp. 187–220.

1937 Coral, M., On the necessary conditions for the minimum of a double integral, Duke Mathematical Journal, vol. III, pp. 585–592.

Finsler, P., Über das Vorkommen definiter und semi-definiter Formen in Scharen quadratische Formen, Commentarii Mathematici Helvitici, vol. IX, pp. 188–192.

1938 Albert, A. A., A quadratic form problem in the calculus of variations, Bulletin of the American Mathematical Society, vol. XLIV, pp. 250–253.

Reid, W. T., A theorem on quadratic forms, Bulletin of the American Mathematical Society, vol. XLIV, pp. 437–440.

Terpstra, F. J., Die Darstellung biquadratischer Formen als Summen von Quadraten mit Anwendung auf die Variationsrechnung, Mathematische Annalen, vol. CXVI, pp. 166–180.

1939 Bliss, G. A., The Calculus of Variations, Multiple Integrals. Mimeographed notes of lectures delivered at the University of Chicago, Spring Quarter 1939.

Graves, L. M., The Weierstrass condition for multiple integral variation problems, Duke Mathematical Journal, vol. V, pp. 656–660.

Landers, A. W., Invariant multiple integrals in the calculus of variations, Contributions to the Calculus of Variations 1938–1941, University of Chicago, pp. 175–207.

Reid, W. T., The Jacobi condition for the double integral problem of the calculus of variations, Duke Mathematical Journal, vol. V, pp. 856–870.

1940 Hestenes, M. R., and McShane, E. J., A theorem on quadratic forms and its application in the calculus of variations, Transactions of the American Mathematical Society, vol. XLVII, pp. 501–512.

1941 Carson, A. B., An analogue of Green's theorem for multiple integral problems of the calculus of variations, Contributions to the Calculus of Variations 1938–1941, University of Chicago, pp. 453–489.

Hestenes, M. R., An analogue of Green's theorem in the calculus of variations, Duke Mathematical Journal, vol. VIII, pp. 300–311.

Dines, L. L., On the mapping of quadratic forms, Bulletin of the American Mathematical Society, vol. XLVII, pp. 494–498.

RATIONAL CURVES DEFINED BY AN ALGEBRAIC CORRESPONDENCE

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1. Introduction. Let $F(\lambda_1, \lambda_2, \lambda_3)$ be an irreducible homogeneous polynomial of degree μ in $\lambda_1, \lambda_2, \lambda_3$, of degree m in λ_1, and of degree n in λ_2. Then the equation

$$F(\lambda_1, \lambda_2, \lambda_3) = 0 \tag{1}$$

establishes an (m, n) algebraic correspondence* between the ratios $(\lambda_1:\lambda_3)$ and $(\lambda_2:\lambda_3)$; that is, any value assigned to the ratio $(\lambda_1:\lambda_3)$ determines n values of the ratio $(\lambda_2:\lambda_3)$, and any value assigned to $(\lambda_2:\lambda_3)$ determines m values of $(\lambda_1:\lambda_3)$. When $(\lambda_1:\lambda_3)$ and $(\lambda_2:\lambda_3)$ are interpreted in homogeneous rectangular coordinates as the two pencils of parallel lines $x_1/x_3=\lambda_1/\lambda_3$ and $x_2/x_3=\lambda_2/\lambda_3$, the locus of the intersection of corresponding lines as established by (1) is the most general algebraic plane curve; the equation of the locus is $F(x_1, x_2, x_3)=0$. Thus, every algebraic plane curve is defined, as is well-known, as the intersection of corresponding lines of two mutually perpendicular pencils of lines, where the correspondence is established by (1).

Let $P_1(\lambda_1:\lambda_3)$ give a point range on the line $A_1(1, 0, 0)$, $A_3(0, 0, 1)$, and let $P_2(\lambda_2:\lambda_3)$ give a point range on the line A_3, $A_2(0, 1, 0)$, where

$$\begin{aligned} P_1 &\equiv \lambda_1 A_3 - \lambda_3 A_1 = (-\lambda_3, 0, \lambda_1) \text{ on } A_1A_3, \text{ and} \\ P_2 &\equiv \lambda_2 A_3 - \lambda_3 A_2 = (0, -\lambda_3, \lambda_2) \text{ on } A_2A_3. \end{aligned} \tag{2}$$

Then the equation of the line joining P_1 to a corresponding point P_2 will be

$$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 = 0;$$

that is, the line coordinates of P_1P_2 (*i.e.*, the parametric line equations of the envelope) are

$$\sigma u_i = \lambda_i, \qquad (i = 1, 2, 3). \tag{3}$$

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* M. Chasles, Comptes Rendus, vol. 58, 1864, p. 1175; W. L. Edge, The Theory of Ruled Surfaces, Cambridge, 1931, p. 9.

Thus it follows that an (m, n) correspondence established by (1) defines the most general algebraic plane curve* whose line equation is $F(u_1, u_2, u_3)=0$. Since F was assumed to be irreducible, then $m+n \geqq \mu$; let $m+n-\mu=\gamma$. Then since F is of degree μ, the envelope established by this correspondence is of class μ; since F is of degree m in u_1 and of degree n in u_2, the lines $[1, 0, 0]$ and $[0, 1, 0]$ are tangents of the envelope and of multiplicities $(n-\gamma)$ and $(m-\gamma)$, respectively.

A non-degenerate line conic is a simple example of a curve defined by an (m, n) correspondence, with $m=n=1$; it is the totality of lines joining corresponding points of a (1, 1) correspondence between two projective, non-perspective ranges of points. The constructions of other particular types of curves by means of (m, n) correspondences have appeared frequently in the journals.

In §2, we obtain parametric line equations of the envelope R of a line joining corresponding points of an (m, n) correspondence established by two involution-ranges of degrees n and m on two fixed lines; every rational plane curve can be so defined. The existence of line singularities is considered in §3, and the Plücker characteristics of R are obtained. It is shown in §4 that every rational curve is the envelope of the Pascal line of a hexagon inscribed in a conic when two of the vertices of the hexagon are allowed to vary by being rational functions of a parameter, and that every algebraic plane curve is the envelope of the Pascal line when the two vertices are related through a general algebraic correspondence.

2. A rational envelope and its parametric equations. If $(t_1 : t_2)$ is the parameter pair of a point on a line and $f_1(t_1, t_2)$ and $f_2(t_1, t_2)$ are relatively prime homogeneous polynomials each of degree n in t_1, t_2, then the set of n points determined by $\lambda_2 f_1(t_1, t_2)=\lambda_1 f_2(t_1, t_2)$ is said to trace out an involution-range† of degree n as λ_1/λ_2 varies. We shall now consider the envelope R of the line P_1P_2 when the (m, n) correspondence between $P_1(\lambda_1 : \lambda_3)$ and $P_2(\lambda_2 : \lambda_3)$ is established by two involution-ranges in the following way.‡ Let

$$\rho\lambda_i = \phi_i(t_1, t_2), \qquad (i = 1, 2, 3),$$

where each ϕ is a homogeneous polynomial of degree μ. Let d_1 be the greatest common divisor of ϕ_2 and ϕ_3, let d_2 be the G. C. D. of ϕ_3 and ϕ_1, and let d_3 be the G. C. D. of ϕ_1 and ϕ_2, so that

$$\begin{aligned} \rho\lambda_1 &= \phi_1 = d_2 d_3 K_1, \\ \rho\lambda_2 &= \phi_2 = d_3 d_1 K_2, \\ \rho\lambda_3 &= \phi_3 = d_1 d_2 K_3, \end{aligned} \tag{4}$$

with each d prime to each other d and each K prime to each other K. Let d_1 be of degree $\mu-m$, and d_2 of degree $\mu-n$, and K_3 of degree γ, so that $m+n-\gamma=\mu$.

* Virgil Snyder, Bulletin of the National Research Council, no. 63, p. 166.

† H. Hilton, Plane Algebraic Curves, Oxford, 1920, p. 375.

‡ The following representation was suggested by Professor W. B. Carver.

Equations (4) represent a rational curve of class μ since its line coordinates are rational functions of degree μ. The correspondence between P_1, P_2 of (2) is established by the ratios,

$$(5) \qquad \frac{\lambda_1}{\lambda_3} = \frac{d_3K_1}{d_1K_3} \equiv \frac{f_1(t_1, t_2)}{f_2(t_1, t_2)}, \qquad \frac{\lambda_2}{\lambda_3} = \frac{d_3K_2}{d_2K_3} \equiv \frac{g_1(t_1, t_2)}{g_2(t_1, t_2)},$$

where the f's are relatively prime homogeneous polynomials each of degree n, and the g's are relatively prime homogeneous polynomials each of degree m. These functions establish involution-ranges of degrees n and m on the lines A_1A_3 and A_2A_3, respectively, and an (m, n) correspondence between the points P_1, P_2, on these two lines. For example, a fixed value of λ_1/λ_3 determines a polynomial equation $\lambda_1 f_2 = \lambda_3 f_1$ of degree n whose n roots $(t_1 : t_2)$ determine n values of λ_2/λ_3. Since the parametric line equations of any rational plane curve are of the form (4), we have the following theorem.

THEOREM 1. *Every rational plane curve is defined by an (m, n) correspondence between the points of two lines in a plane where the correspondence is established by the two involution-ranges* (5).

It is apparent, from (2) and (5), that the point A_3 corresponds to the parameter pair $(\lambda_1 : \lambda_3) = (1 : 0)$ on A_1A_3 and $(\lambda_2 : \lambda_3) = (1 : 0)$ on A_2A_3. Consequently, the point A_3 is a self-corresponding point of multiplicity γ in the correspondence and its parameter pairs $(t_1 : t_2)$ on each of the two lines are the γ roots of $K_3(t_1, t_2) = 0$. It is apparent from (4) that the lines $A_1A_3 = [0, 1, 0]$ and $A_2A_3 = [1, 0, 0]$ are tangents of multiplicities $(m - \gamma)$ and $(n - \gamma)$, respectively, with parameter pairs $(t_1 : t_2)$ which are the roots of $d_2 = 0$ and $d_1 = 0$.

3. Other multiple tangents of the rational curve R and its Plücker characteristics. It is possible to obtain* immediately the Plücker characteristics of the envelope when the correspondence is established by (1) when it is assumed that the only line singularities are A_1A_3 and A_2A_3. A rational curve of class $\mu = 3$, for example, can have at most one bitangent; if A_1A_3 is a bitangent, the line A_2A_3 can be at most an ordinary tangent. When the value of μ is larger, however, it will be seen that R can have other multiple tangents.

The deficiency of a curve of order ν, class μ, with δ nodes, κ cusps, i stationary tangents, and τ bitangents is

$$(6) \qquad p = \tfrac{1}{2}(\nu - 1)(\nu - 2) - \delta - \kappa = \tfrac{1}{2}(\mu - 1)(\mu - 2) - \tau - i.$$

For the curve R we have $\mu = m + n - \gamma$ and $p = 0$; consequently,

* A. Emch, The symmetric (n, n)-correspondence and some geometric applications, American Jour. of Math., vol. 54, no. 2, 1932, pp. 285–292; Hilton, *loc. cit.*, pp. 372–374.

$$\tau + i = \tfrac{1}{2}(m + n - \gamma - 1)(m + n - \gamma - 2). \tag{7}$$

The two known multiple tangents of R account for*

$$\tau_1 = \tfrac{1}{2}[(m - \gamma)(m - \gamma - 1) + (n - \gamma)(n - \gamma - 1)] \tag{8}$$

bitangents and stationary tangents. When the roots of $d_2=0$ and $d_1=0$ are distinct the corresponding contact points of R on A_iA_3 will be distinct ordinarily, and these two tangents will account for τ_1 bitangents and no stationary tangents. Then it can be seen from (7), (8) that the remaining† number to account for is

$$(m - 1)(n - 1) - \tfrac{1}{2}\gamma(\gamma - 1). \tag{9}$$

Ordinarily, the variable point P_1 will occupy any fixed position A on A_1A_3 for n different parameter pairs $(t_1:t_2)$ and, consequently, the n corresponding positions of P_2 on A_2A_3 are usually distinct. If two of the parameter pairs for which P_1 is at A $(A \neq A_3)$ are $t_1:t_2=x:z$ and $t_1:t_2=y:z$, with $x \neq y$, and also if P_2 is at B on A_2A_3 for these two ratios, then the line AB is a bitangent‡ of R. From (5), this relationship will exist for certain pairs of solutions, $(x:z)$ and $(y:z)$ of the equations,

$$\begin{cases} f_1(x, z)f_2(y, z) - f_1(y, z)f_2(x, z) = 0, \\ g_1(x, z)g_2(y, z) - g_1(y, z)g_2(x, z) = 0. \end{cases} \tag{10}$$

In these equations, since the terms of highest degree in x, y cancel, the factor z can be removed from each, and since any pair $x=y$ is a formal solution the factor $(x-y)$ can be removed from each. Then the resulting pair of equations,

$$\theta_1(x, y, z) = 0 \quad \text{and} \quad \theta_2(x, y, z) = 0$$

will be of degrees $(2n-2)$ and $(2m-2)$ in x, y, z. But θ_1 is of degree $(n-1)$ in x and of degree $(n-1)$ in y so that $\theta_1=0$, as a curve, has the points (1, 0, 0) and (0, 1, 0) as points of multiplicity $(n-1)$. Similarly the curve $\theta_2=0$ has these same two points as points of multiplicity $(m-1)$. In general these two multiple points account for $2(m-1)(n-1)$ solutions of (10) which do not determine bitangents. Furthermore, the γ roots of $K_3=0$ are the parameters of A_3 on each line and they account for $\gamma(\gamma-1)$ solutions of (10). Since each of the equations $\theta_i=0$ is symmmetrical in x and y and the solutions (x, y, z) and (y, x, z) give parameters of the same bitangent, the remaining $2(m-1)(n-1)-\gamma(\gamma-1)$ solutions of (10) determine

$$(m - 1)(n - 1) - \tfrac{1}{2}\gamma(\gamma - 1) \tag{11}$$

bitangents of R. By comparing (9) and (11), it is seen that the solutions of (10)

* Any k-tuple tangent with α distinct points of contact is equivalent to $[\frac{1}{2}k(k-3)+\alpha]$ bitangents and $(k-\alpha)$ stationary tangents, a total of $\frac{1}{2}k(k-1)$. See Hilton, *loc. cit.*, p. 116.

† In an $(m, 1)$ or a $(1, n)$ correspondence, only one of the two lines is a multiple tangent, and it accounts for all of the bitangents and stationary tangents of the curve.

‡ The line AB would be a stationary tangent only if the coefficients of f_i and g_i were such that the two pairs $(x:z)$ and $(y:z)$ give the same contact point of R on AB.

determine the remaining bitangents and stationary tangents.

If of the solutions there is a pair such that $x=y$, the two points of tangency of R on AB will be coincident, and the line AB will be a stationary tangent of R. If the coefficients of f_i and g_i satisfy the conditions necessary to produce β pairs of solutions $x=y$, the solutions of (10) will determine

$$\tau_2 = (m-1)(n-1) - \beta - \tfrac{1}{2}\gamma(\gamma-1)$$

bitangents and β stationary tangents. Consequently the number τ of bitangents and the number i of stationary tangents of R has been determined to be

$$\tau = \tau_1 + \tau_2 = \tfrac{1}{2}(m+n-\gamma-1)(m+n-\gamma-2) - \beta$$

and

$$i = \beta.$$

The remaining Plücker characteristics of the curve can now be determined and are the following:

$$\nu = 2(m+n-\gamma-1) - \beta,$$
$$\kappa = 3(m+n-\gamma-2) - 2\beta,$$
$$\delta = 2(m+n-\gamma-2)(m+n-\gamma-\beta-3) + \tfrac{1}{2}\beta(\beta+3).$$

4. The curve R as the envelope of a Pascal line. If six points are selected on a conic and these points are taken as the vertices of an ordered hexagon, then the three pairs of opposite sides meet in three collinear points, P_1, P_2, and P_3. This is Pascal's theorem; the points P_i $(i=1, 2, 3)$ are called the Pascal points and the line through them is called the Pascal line.

If five of the vertices are fixed and the sixth is then allowed to move around the conic, the Pascal line will revolve about a fixed Pascal point. Certain other interesting loci will be generated, however, and they have been investigated by Bunch.* When two of the vertices, say Q_1 and Q_2 are related by having parameters on the conic which are functions of a common parameter pair $(t_1:t_2)$, the Pascal line will envelop a curve as $(t_1:t_2)$ varies. This curve will be merely the fixed Pascal point when Q_1 and Q_2 are opposite vertices of the ordered hexagon. We shall consider in this section the envelope which is generated when Q_1 and Q_2 are not opposite vertices of the hexagon.

Any non-degenerate conic through any four fixed points B_1, B_2, B_3, and B_4, no three collinear, is the locus of a point P which moves so that the cross ratio of the four lines PB_i is a constant k $(k\neq 0, 1, \infty)$. The four points B_i may be projected into the four fixed points $A_1(1, 0, 0)$, $A_2(0, 1, 0)$, $A_3(0, 0, 1)$ and $A_4(1, 1, 1)$, and if the cross ratio of the lines PA_i joining any point P on the conic to the A_i is equal to a constant k, the equation of the conic is

$$x_2x_3 - kx_1x_3 + (k-1)x_1x_2 = 0. \tag{12}$$

* W. H. Bunch, this MONTHLY, vol. 40, 1933, pp. 251–260.

Then it follows that the parametric point equations of any conic may be written as follows:

$$\begin{aligned} \rho x_1 &= s_2(s_1 + s_2), \\ \rho x_2 &= k s_1 s_2, \\ \rho x_3 &= (k-1)s_1(s_1 + s_2). \end{aligned} \tag{13}$$

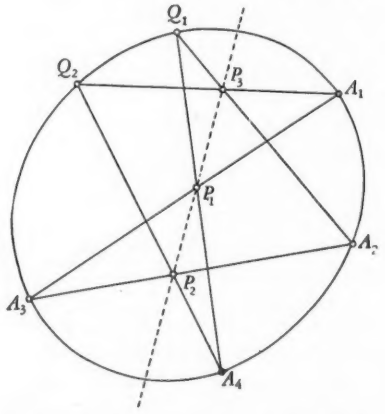
The three conics of the pencil which correspond to $k=0,\ 1,\ \infty$ are degenerate. The parameter pairs $(s_1:s_2)$ corresponding to A_1, A_2, A_3, and A_4 on the conic are $(0:1)$, $(1:-1)$, $(1:0)$, and $(1:k-1)$, respectively. Now let the variable points Q_1 and Q_2 on the conic correspond to the parameter pairs $(s_1^{(1)}:s_2^{(1)})$ and $(s_1^{(2)}:s_2^{(2)})$, respectively. We establish an (m, n) correspondence between Q_1 and Q_2 by letting

$$\frac{s_1^{(1)}}{s_2^{(1)}} = \frac{f_1(t_1, t_2)}{f_2(t_1, t_2)} \quad \text{and} \quad \frac{s_1^{(2)}}{s_2^{(2)}} = \frac{g_1(t_1, t_2) + g_2(t_1, t_2)}{-\,g_2(t_1, t_2)} \tag{14}$$

where the f's are relatively prime homogeneous polynomials each of degree n, and the g's are relatively prime homogeneous polynomials each of degree m, as in §2. We shall now prove the following theorem.

THEOREM 2. *Every rational plane curve is defined as the envelope of the Pascal line of a hexagon inscribed in a conic by an (m, n) corresponding between two variable vertices, where the correspondence is established as in* (14).

In order to show the relationship between the envelope of the Pascal line and the locus R of §2, it is convenient to choose the vertices of the ordered hexagon inscribed in the conic to be A_1, A_3, A_2, Q_1, A_4, Q_2, respectively. Corresponding



to each point Q there is *first* (see the figure) a Pascal point P_1 on A_1A_3 and, *second*, there are n values of $(t_1:t_2)$ and therefore n positions of Q_2; these n positions

of Q_2 determine n Pascal points P_2 on A_2A_3. Consequently, the (m, n) correspondence between Q_1 and Q_2 on the conic established by (14) also establishes an (m, n) correspondence between the Pascal points P_1 and P_2 of A_1A_3 and A_2A_3. With the equations (13) of the conic, the parameters (14) of the Q's, and the parameters (2) of the P's, the coordinates of the points P can be determined to be $P_1(-\lambda_3, 0, \lambda_1)$ and $P_2(0, -\lambda_3, \lambda_2)$, where as in (5)

$$\lambda_1/\lambda_3 = f_1/f_2 \quad \text{and} \quad \lambda_2/\lambda_3 = g_1/g_2.$$

Then as the Q's move along the conic, the Pascal points P_1, P_2 move along the two fixed lines exactly as in §2, and the Pascal line will envelope any rational plane curve.

Of the 60 ordered hexagons whose vertices are the preceding six points in some order, there are 12 in which the Q's are opposite vertices. The locus which corresponds to each of the remaining 48 orders is essentially the same as the one for the ordered hexagon described above. In each of these 48 cases there are at least three fixed consecutive vertices, say A, B, C. The (m, n) correspondence between the Q's which is determined by (14) establishes an (m, n) correspondence between the Pascal points on AB and BC provided there are exactly three fixed consecutive vertices. When there are four fixed consecutive vertices, A, B, C, D, the (m, n) correspondence will be established between the Pascal points of AB and CD. The (m, n) correspondence will be established by parameter pairs which are rational homogeneous functions of degrees n and m, respectively.

It can be noted that *every* algebraic curve, whether rational or not, can be defined as the envelope of the Pascal line of the hexagon A_1, A_3, A_2, Q_1, A_4, Q_2, by means of an (m, n) correspondence between Q_1 and Q_2 on the conic when the other four vertices are fixed. Let $F(\lambda_1, \lambda_2, \lambda_3) = 0$ be the line equation of the curve where F is of degree m in λ_1, of degree n in λ_2, and μ in λ_1, λ_2, λ_3. Then as in §2 this equation establishes an (m, n) correspondence between the point ranges $P_1 = \lambda_1 A_3 - \lambda_3 A_1$ and the point range $P_2 = \lambda_2 A_3 - \lambda_3 A_2$. When the parameters of Q_1 and Q_2 on the conic (13) are chosen to be $(\lambda_1 : \lambda_3)$ and $(\lambda_2 + \lambda_3 : -\lambda_3)$, the Pascal line will envelop any algebraic plane curve $F(\lambda_1, \lambda_2, \lambda_3) = 0$.

THE NUMERICAL INTEGRATION OF $y''+g(x)y=f(x)$

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This note presents a simple procedure for the numerical integration of the linear differential equation

$$y'' + g(x)y = f(x), \tag{1}$$

in which primes denote differentiation with respect to x, and g and f are known functions.

1. Formal derivation. To derive a suitable formula for our purpose we differentiate Stirling's central difference interpolation formula twice, set the argument equal to the central value, and obtain (using Sheppard's central difference operator δ)

$$\begin{aligned} y'' &= \frac{1}{h^2}\left[\, 2\,\sinh^{-1}\frac{\delta}{2}\,\right]^2 y \\ &= \frac{1}{h^2}\left[\, \delta^2 y - \frac{\delta^4 y}{12} + \frac{\delta^6 y}{90} - \cdots \right], \end{aligned} \tag{2}$$

in which h denotes the length of the interval between equally spaced values of x. Next we operate on (1) and (2) with the operator δ^2, which gives

$$\delta^2 y'' + \delta^2[g(x)y] = \delta^2 f(x) \tag{3}$$

and

$$\delta^2 y'' = \frac{1}{h^2}\left[\, \delta^4 y - \frac{\delta^6 y}{12} + \cdots \right]. \tag{4}$$

From equations (1), (2), (3), and (4) we eliminate the three quantities y'', $\delta^2 y''$, and $\delta^4 y$.

The resulting equation is

$$\delta^2\left[\, y + \frac{h^2 g(x)}{12}\, y \right] + h^2 g(x) y = h^2 f(x) + \frac{h^2\delta^2 f(x)}{12} - \frac{\delta^6 y}{240} + \cdots . \tag{5}$$

The substitutions

$$\left[\, 1 + \frac{h^2 g(x)}{12} \right] y = z, \quad \frac{h^2 g(x)}{1 + \dfrac{h^2 g(x)}{12}} = G(x),$$

and

$$h^2 f(x) + \frac{h^2\delta^2 f(x)}{12} = F(x),$$

carry this equation into

$$\delta^2 z + G(x)z = F(x) - \frac{\delta^6 y}{240} + \cdots . \tag{6}$$

Now if the interval is small enough so that the term $\delta^6 y/240$ is negligible, equation (6) reduces to a simple linear difference equation of the second order.

2. Tabular arrangement. The procedure in solving (6) numerically is obvious. First we select a suitable value for h, then tabulate the values of $G(x)$ and $F(x)$ according to the scheme shown below:

x	$F(x)$	$G(x)$	z	Δz	$\Delta^2 z$
x_0	F_0	G_0	z_0		
x_1	F_1	G_1	z_1	Δz_0	
x_2	F_2	G_2	z_2	Δz_1	$\Delta^2 z_0 = \delta^2 z_1$
x_3	F_3	G_3			
x_4	F_4	G_4			

The first two values z_0 and z_1 are assumed to be known, and from them we get $\Delta z_0 = z_1 - z_0$. Then from (6) we have

$$\Delta^2 z_0 = \delta^2 z_1 = -G_1 z_1 + F_1,$$

and we complete the line for x_2 with the formulas,

$$\Delta z_1 = \Delta z_0 + \Delta^2 z_0,$$
$$z_2 = z_1 + \Delta z_1.$$

The integration proceeds by repetitions of this process. When it is completed, the values of y, or at least as many of them as are desired, are obtained from

$$y = \frac{z}{1 + \dfrac{h^2 g(x)}{12}} .$$

If the general solution of (1) is required it may be obtained by solving

$$\delta^2 u + G(x)u = 0, \qquad u_0 = 0, \qquad u_1 = h,$$
$$\delta^2 v + G(x)v = 0, \qquad v_0 = v_1 = 1,$$

and

$$\delta^2 w + G(x)w = F(x), \qquad w_0 = w_1 = 0.$$

The general solution of (6) is then

$$z = C_1 u + C_2 v = w,$$

and from z we readily obtain y.

3. Example. Consider the differential equation

$$y'' + \frac{3+4x}{16x^2}\,y = 0, \quad y = 1, \quad y' = 0, \quad \text{at} \quad x = 1.$$

Since $f(x)=0$ we need calculate only $G(x)$, which for $h=0.1$ proves to be

$$G(x) = \frac{12(3+4x)}{19200x^2 + (3+4x)}.$$

The values of $G(x)$ are calculated from this expression and tabulated as shown below. We find by a few terms of Taylor's series that $y_1=0.997911$, so that $z_0=1.000364$, $z_1=0.998228$. Everything needful is now known and the calculation is completed to $x=2.0$ as follows:

x	$G(x)$	z	Δz	$\Delta^2 z$	y
1.0	.0043734	1.000364			1.000000
1.1	38211	.998228	−2136		
1.2	33845	.992278	−5950	−3814	
1.3	30318	.982970	−9308	−3358	
1.4	27417	.970682	−12288	−2980	
1.5	24995	.955733	−14949	−2661	
1.6	22945	.938395	−17338	−2389	
1.7	21190	.918904	−19491	−2153	
1.8	19673	.897466	−21438	−1947	
1.9	18349	.874262	−23204	−1766	
2.0	17185	.849454	−24808	−1604	.849333

The equation chosen above can be solved explicitly, and the particular solution desired proves to be

$$y = x^{1/4}[\cos(x^{1/2} - 1) - 0.5 \sin(x^{1/2} - 1)].$$

The substitution of $x=2$ in this expression gives $y=0.8493293$ which differs only by about four units in the sixth decimal place from the value found by numerical ntegration.

AN INVERSE PROBLEM IN CORRELATION THEORY

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1. Introduction. The present problem, more of mathematical interest than practical, grew out of a question which may confront the instructor in statistics. He might, in concocting exercises in the computation of multiple and partial correlation coefficients, give the students a square array of numbers r_{ij} such that $r_{ij}=r_{ji}$, $r_{ij}^2 \leqq 1$, $r_{ii}=1$. If these have not actually been computed from data (observed or manufactured), *how can he be sure of the existence of a set of data which would give rise to such hypothetical correlation coefficients* r_{ij}? Under the restriction, usually desired in such exercises, that the determinant of the array be not zero (non-singular case), a simple criterion is found in Corollary 4 below. Otherwise, the general case (possibly singular) is considered. Sections 2 and 3 may be of some general interest inasmuch as the language of matrices is used to exhibit the dependence of the coefficients r_{ij} on the data and to state necessary conditions on these coefficients. In the course of proving the existence theorem in section 4, particular solutions are constructed and the most general solution is characterized. Some corollaries are drawn in section 5.

2. Matrix formulation. m sets of measurements on each of n variables may be written in the form of a matrix,*

$$Y^{m\times n} = \begin{pmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ y_{m1} & y_{m2} & \cdots & y_{mn} \end{pmatrix}, \tag{1}$$

where y_{ij} denotes the ith observation on the jth variable. A matrix (1) will be referred to as *a statistics problem.*

The present approach employs the concepts of unitary vectors and orthogonal vectors. A vector, ξ, that is, a column matrix, is unitary if $\xi'\xi=1$. Two vectors, ξ, η, are orthogonal if $\xi'\eta=0$. At this point it is convenient to define the unitary vector $\psi^{m\times 1}$ all of whose elements are $m^{-1/2}$. Then ψ' applied to the left of a matrix gives a row matrix equal to $m^{-1/2}$ times the sum of the rows of the original. $M=\psi\psi'$ is an $m\times m$ matrix all of whose elements are m^{-1}; applied to the left of a matrix it yields a new one all of whose rows are the same, and the mean of the rows of the old. *The matrix of means* is defined as

$$\overline{Y}^{m\times n} = MY. \tag{2}$$

* Some of the notation differs from standard practice in order that the following scheme may be rigidly followed: Upper case Latin letters denote matrices; lower case Latin, scalars; lower case Greek (except the Kronecker delta), vectors. A superscript $s\times t$ attached to a matrix signifies that it has s rows and t columns. *All matrices and scalars are real.* A' means the transpose of A.

The matrix of deviations is defined to be

$$X^{m\times n} = Y - \overline{Y}, \tag{3}$$

or

$$X = DY, \quad \text{where} \quad D^{m\times m} = I^{m\times m} - \psi\psi',$$

and I is the identity matrix. We note

$$\psi' X = 0, \quad \text{since} \quad \psi' D = \psi' - \psi' = 0.$$

In order to obtain unitary vectors from the columns of X, we define $S^{n\times n}$ as that diagonal matrix whose jth diagonal element is the non-negative square root of the jth diagonal element of $X'X$, that is, $s_{jj}=(\xi_j' \xi_j)^{1/2}$, where ξ_j is the jth column of X. The diagonal matrix S^{-1} exists unless one of the columns of Y has all identical elements, and we now exclude this trivial case. *The matrix of normalized deviations** is now defined as

$$Z^{m\times n} = XS^{-1}. \tag{4}$$

We note again,

$$\psi' Z = \psi' XS^{-1} = 0. \tag{5}$$

Finally, *the correlation matrix* is defined to be

$$R^{n\times n} = Z'Z, \tag{6}$$

in agreement with the usual definition of r_{ij}, the element in the ith row and the jth column. The matrix R is thus a function of the statistics problem (1),

$$R = F(Y) = S^{-1}Y'D'DYS^{-1}. \tag{7}$$

3. Necessary conditions. It will be convenient to refer to the following as

Conditions $(\mathcal{C})$: $(\mathcal{C}-1)$: *R is symmetric.*
$(\mathcal{C}-2)$: *The diagonal elements of R are unity.*
$(\mathcal{C}-3)$: *R is positive, i.e., rank R = index R.*

A test for $(\mathcal{C}-3)$ if $(\mathcal{C}-1)$, $(\mathcal{C}-2)$ hold is found in Corollary 3 below.

LEMMA 1. *The conditions $(\mathcal{C})$ are necessary if $R=F(Y)$.*

The necessity of $(\mathcal{C}-1)$ follows from the rule for the transpose of a product, $(AB)'=B'A'$, applied to (6), that of $(\mathcal{C}-2)$ from the unitary property of the columns of Z. To prove $(\mathcal{C}-3)$, we note first that the determinants of the principal minors of R are non-negative, since† they are Gramian determinants formed

* The matrix of deviations in standard units is simply $T=m^{1/2}Z$, but we shall not need this in the present discussion. Also, the diagonal elements of S are $s_{jj}=m^{1/2}s_j$, where s_j is the standard deviation of the jth variable.

† Courant-Hilbert, Methoden der mathematischen Physik, Berlin, 1931, vol. 1, ch. 1.

from certain columns of Z. This in turn is sufficient* to insure that R is positive.

The lemma suggests the question, are the conditions ($\mathcal{C}$) sufficient that $R=F(Y)$? The theorem below contains the answer. To prove it we need another

LEMMA 2. *Rank* $R=$*rank* $Z<m$.

Since the rank of a product cannot exceed the rank of a factor, $p(R)\leqq p(Z)$, where $p(A)$ denotes rank A. But we can pick $p(Z)$ independent columns from Z, and their Gramian matrix appears as a non-singular† minor in R. Hence $p(R)\geqq p(Z)$. This proves the equation in the lemma; the inequality follows from (5) which implies that the m rows of Z are dependent.

4. The existence theorem: *Given any matrix* $R^{n\times n}$ *of rank* p *and satisfying the conditions* ($\mathcal{C}$), *consider the possibility of a solution* $Y^{m\times n}$ *of the equation* $R=F(Y)$. *There is no solution with* $m\leqq p$, *but there are* ∞^q *solutions for each* $m>p$, *with* $q=2n+p(2m-p-3)/2$.

The negative aspect of the theorem follows immediately from our Lemma 2. To prove the positive statement we transform the problem. By a finite number of rational operations on the elements of R, and n square root extractions, a non-singular matrix $P^{n\times n}$ can‡ be constructed such that

$$P'RP = E,$$

a matrix whose first p diagonal elements are unity, and all other elements zero. A 1:1 correspondence is established between matrices $Z^{m\times n}$ and $V^{m\times n}$ by the equations

$$V = ZP, \qquad Z = VP^{-1}. \tag{8}$$

If

$$Z'Z = R \quad \text{and} \quad \psi'Z = 0, \tag{9}$$

then

$$V'V = E \quad \text{and} \quad \psi'V = 0, \tag{10}$$

and conversely.

Contemplating the transformed problem (10), we see that the first p columns of V, say $\kappa_1, \kappa_2, \cdots, \kappa_p$, must be unitary and mutually orthogonal, $\kappa_i'\kappa_j=\delta_{ij}$, and the remaining columns all zero. Hence if we write V as a composite matrix

$$V = (K^{m\times p}0^{m\times(n-p)}), \tag{11}$$

where K is the matrix whose columns are $\kappa_1, \kappa_2, \cdots, \kappa_p$, then

$$K'K = I^{p\times p}, \qquad \psi'K = 0. \tag{12}$$

* L. E. Dickson, Modern Algebraic Theories, N. Y., 1930, ch. 4.

† Courant-Hilbert, *loc. cit.*

‡ Dickson, *loc. cit.*

Consider the $(m-1)$-dimensional subspace $\mathcal{L}$ of m-space composed of vectors κ orthogonal to ψ, $\psi'\kappa=0$. A unitary orthogonal basis for $\mathcal{L}$ may be constructed as follows: Define the matrix $L^{m\times(m-1)}$ as that obtained by bordering the identity matrix of order $m-1$ with a row of $m-1$ elements each of which is -1. Obviously $\psi'L=0$. The $m-1$ independent columns of L form a basis for $\mathcal{L}$. To achieve a unitary orthogonal basis we can select* suitable linear combinations of these columns by a finite number of non-tentative steps (Schmidt procedure). If the coefficients in each linear combination are written as a column in a square matrix C of order $m-1$, then the columns of

$$U^{m\times(m-1)} = LC$$

form the desired basis for $\mathcal{L}$,

$$U'U = I \quad \text{and} \quad \psi'U = \psi'LC = 0.$$

Particular solutions of the problem (12) can now be obtained by choosing any p columns of U for the columns of K. The most general solution is of the type

$$K = UW^{(m-1)\times p},$$

where

$$W'W = I.$$

First, note that any such K is a solution,

$$K'K = W'U'UW = W'W = I, \qquad \psi'K = \psi'UW = 0;$$

secondly, that any solution is of this form: Suppose K satisfies the equations (12). Then its columns are in the space $\mathcal{L}$ and hence are linear combinations of the columns of U, $K=UW$. Furthermore, $I=K'K=W'U'UW=W'W$.

The number of distinct solutions for K is the same as the number of distinct matrices $W^{(m-1)\times p}$ for which $W'W=I$, since there is a 1:1 correspondence between possible K and W, given by

$$K = UW, \qquad W = U'K.$$

The restriction $W'W=I^{p\times p}$ puts p^2 conditions on the $p(m-1)$ elements of W. But since $W'W$ is automatically symmetric, only $p(p+1)/2$ of these conditions are independent. Hence the number of solutions for K is ∞^{q_1} where $q_1=p(m-1)-p(p+1)/2$. V is then uniquely determined by (11) and Z by (8). The n diagonal elements of S can now be chosen as arbitrary positive numbers, after which X is determined by

$$X = ZS.$$

Finally, the n elements of the equal rows of $\overline{Y}$ may be chosen arbitrarily, determining

* Courant-Hilbert, *loc. cit.*

$$Y = X + \overline{Y}.$$

The number of solutions for Y is thus ∞^q with $q = q_1 + 2n$.

5. Some corollaries. Corollaries of some inherent algebraic interest may now be deduced from our theorem by noting necessary conditions other than $(\mathcal{C})$ for $R = F(Y)$ and then asserting these as properties of any matrix satisfying $(\mathcal{C})$. We illustrate by using familiar properties of the multiple and partial correlation coefficients for the problem (1).

If η_1, η_2 are any vectors of m elements, then $\xi_1 = D\eta_1$, $\xi_2 = D\eta_2$ are the respective deviations from the means, and the correlation coefficient between the vectors is $r(\eta_1, \eta_2) = \xi_1' \xi_2 / (\xi_1' \xi_1 \cdot \xi_2' \xi_2)^{1/2}$, providing the denominator does not vanish. The Schwartzian inequality yields immediately $r^2 \leqq 1$. Since the coefficients of multiple and partial correlation may† be defined as $r(\eta_1, \eta_2)$, where η_1 and η_2 are certain linear combinations of the columns of Y, they satisfy the relation $0 \leqq r^2 \leqq 1$ whenever they are defined. Using well-known formulas for these coefficients, we get inequalities on determinants of minors of R of order $n-1$ and n. Further inequalities may be obtained by the device of considering a "sub-problem Y^*" of (1), where the columns of Y^* are a subset of the columns of Y. Then $R^* = F(Y^*)$ is a principal minor of R, say of order n_1, and the corresponding inequalities involve determinants of minors of R of order $n_1 - 1$ and n_1.

We now define *bordering sequences*

$$d_1, d_2, \cdots, d_n \tag{13}$$

as follows: Pick any principal diagonal element R_1^* of R, then any 2×2 principal minor R_2^* containing R_1^*, then any 3×3 principal minor R_3^* containing R_2^*, etc. A sequence (13) consists of the determinants of a sequence $R_1^*, R_2^*, \cdots, R$. In terms of

Conditions $(\mathcal{D})$: *Every bordering sequence* (13) *formed from* R *has the property*

$$1 = d_1 \geqq d_2 \geqq \cdots \geqq d_n \geqq 0,$$

we may state

LEMMA 3. *The conditions* $(\mathcal{D})$ *are necessary if* $R = F(Y)$.

That all determinants of principal minors are non-negative was noted in connection with Lemma 1. If $d_{j-1} > 0$, then $1 - d_j/d_{j-1}$ is the square of a multiple correlation coefficient r for a sub-problem Y^*, and since $r^2 \geqq 0$, $d_{j-1} \geqq d_j$. If $d_{j-1} = 0$, then $d_j = 0$, since d_{j-1} is the Gramian determinant of a subset of those columns of Z which lead to the Gramian determinant d_j. Hence in every case $d_{j-1} \geqq d_j$.

COROLLARY 1. *Any matrix* R *satisfying the conditions* $(\mathcal{C})$ *satisfies the conditions* $(\mathcal{D})$.

† Yule and Kendall, An Introduction to the Theory of Statistics, London, 1937, pp. 277 and 266.

By similar† reasoning applied to the partial correlation coefficients of subproblems Y^*, we are led to the following: Let R^* be any principal minor of R (including R as a special case). Let d_{ij}^* be the determinant of the minor of R obtained by striking out the ith row and jth column of R.*

Conditions ($\mathcal{E}$): *For every R^* and every $i, j \leqq$ order R,**

$$d_{ij}^{*2} \leqq d_{ii}^* \, d_{jj}^*.$$

LEMMA 4. *The conditions* ($\mathcal{E}$) *are necessary if* $R = F(Y)$.

COROLLARY 2. *Any matrix R satisfying the conditions* ($\mathcal{C}$) *satisfies the conditions* ($\mathcal{E}$).

We conclude with a procedure for answering the question raised in section 1:

COROLLARY 3. *A method for testing whether or not a given symmetric matrix R with main diagonal elements unity is a possible correlation matrix, $R = F(Y)$, is the following: Start constructing a bordering sequence* (13). *Suppose a stage is reached where $d_1, d_2, \cdots, d_p$ are all positive. If a d_{p+1} is picked such that $d_{p+1} < 0$, R is ruled out. If the chosen $d_{p+1} = 0$, others have to be evaluated.‡ If all $d_{p+1} = 0$ the test is always concluded at the next step: if some $d_{p+2} \neq 0$, R is ruled out; if all $d_{p+2} = 0$, R is a possible correlation matrix of rank p.*

An important special case of this is

COROLLARY 4. *Necessary and sufficient conditions that a symmetric matrix R with main diagonal elements unity be a non-singular correlation matrix are the $n-1$ conditions that the determinants*

$$\begin{vmatrix} 1 & r_{12} \\ r_{21} & 1 \end{vmatrix}, \begin{vmatrix} 1 & r_{12} & r_{13} \\ r_{21} & 1 & r_{23} \\ r_{31} & r_{32} & 1 \end{vmatrix}, \cdots, |R|$$

be all positive.

It is clear that the determinants in Corollary 4 may be replaced by those from any other bordering sequence (13). This again implies that if R is to be a non-singular correlation matrix all off-diagonal elements must be numerically less than unity.

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† We now use the second of the inequalities $0 \leqq r^2 \leqq 1$. A separate argument for Lemma 4 must again be made in the case where d_{ii}^* or $d_{jj}^* = 0$.

‡ If $p+1 =$ order R, there is of course a unique $d_{p+1} = |R|$. Then if $|R| = 0$, $R = F(Y)$ and rank $R = p$.

MEANS AND ENDS IN MATHEMATICS

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It seems to the writer that the outstanding weakness of mathematical instruction is its failure to make clear to the students the purpose, the aims, of various parts of the curricula, and its failure to show how the *means* used in mathematics follow naturally from those *ends*. In other words, while our courses are organized more or less logically from the postulational point of view, they are not so organized from the point of view more important to the student, that of meaning and purpose.

Although trigonometry or analytic geometry would exemplify these statements, let us use instead the pervasive subject of algebra. Much of the prevalent fear and dislike of algebra arises from our failure to make clear to students and to prospective students the *aims* of algebra, and from the inevitable resultant feeling that algebra is artificial, unnatural. A brief discussion of the purposes of algebra may help to substantiate this idea.

First, let us make a distinction between "basic arithmetic" and "applied arithmetic" and between "basic algebra" and "applied algebra." The purpose of basic arithmetic can be simply stated: it is to develop ability in performing the four fundamental operations, on whole numbers and on fractions. (This omits reference to taking roots, and to irrational numbers in general.) It is important for our discussion to note that, as far as the better students are concerned, basic arithmetic completely achieves its purpose. What, then, is the purpose of basic algebra? It must be different from that of basic arithmetic. Our students often think that the aim of algebra is to deal with a *new* kind of number, the literal number. But we know that letters are not a new kind of number, in fact are not numbers at all, but new names for numbers. What new purpose creates the need for these new names?

Sometimes our students think that the aim of algebra is to solve problems and find unknown numbers. (I have often been told by students with two or more years of college mathematics that algebra deals with unknown numbers, arithmetic with known numbers, as if the answer to an arithmetic problem were more "known" than that of an algebra problem!) They, of course, are thinking of applied algebra. This is a very important branch, but before one can have applied algebra, he must have some basic algebra to apply.

In any text we find a clue to the answer to our query. We see the equation $x+x=2x$, which is not a statement about unknown numbers, but about *all* numbers, just as "Man is born to die," though it uses the singular form of the noun, is a statement about all men. Most of algebra is, in a sense, an elaboration of the distributive law, $x(y+z)=xy+xz$, which is a truth about all triples of numbers (real or complex, at least). The purpose of basic algebra is to discover such truths, and to state them in a convenient language. Stated postulationally, one's aim is to discover certain statements, and deduce others from these, such that the totality forms a system suitable for "ordinary" applications.

Not even in college courses do students ordinarily learn to look upon "identities" as actual truths about numbers. This can be shown by a simple test. A group which apparently has mastered $4x+8x=12x$, i.e., which "knows" that it is an identity, is asked "If, for some reason, you wished to add 5 times 17 to 3 times 17 without first multiplying, what would you get?" A large majority of the answers are "8 times 34." Other examples could be given, but the fact is too well known to require further illustration.

A student who understands that basic arithmetic has achieved its aim, and that algebra has this new aim, hardly touched before, of discovering the general truths or "laws" of numbers sees readily the *need* for new words "x," "y," etc., in algebra. He sees how clumsy it is to write or state in English that "If I take any two numbers and add them together, and multiply the result by a third, $\cdots$," i.e., the distributive law. Understanding the need for new words eliminates the apparent artificiality of algebra.

When we remember that it took thousands of years for mankind to discover such algebraic truths, and to formulate them in a convenient language, we become skeptical of the speed with which we expect our students to grasp these truths, and their formulation. What reason is there to expect such a grasp at all, if we do not state over and over the purpose of algebra, and of its identities, if we do not train our students to read the abstract language of algebra, with time and patience comparable to that which we use in their English training? If we do not help them, over a period of months or years, to assimilate the *truth* of equations such as $x(y+z)=xy+xz$, rather than to look upon it as a *problem* in multiplication, when read from left to right, or in factoring, when read from right to left?

A final conclusion from our analysis of the purpose of algebra is: algebra and arithmetic should be integrated. In no other subject do we spend eight years teaching specific facts, without ever hinting at the general truths underlying them. In no other field do we so artificially divorce theory and practice, when the goal is to integrate them. The language of algebra, when introduced in a week, with no comprehension of its purpose, is not understood by our students, and they fall back on memorizing. If introduced gradually, starting in the early grades, with emphasis always on the *meaning* of statements, it is not difficult, as the writer has verified. A ten-year old boy accepts the statements that "α is a new word, which means 'every number'," and "α plus α is always the same as 2 times α" as interesting and meaningful, and he can verify the latter for the numbers in his range of experience. Algebra, to him, has a purpose.

This note has used algebra as an example of its thesis. In other fields of mathematics the emphasis on meaning and purpose does not, in the writer's opinion, imply as drastic a reorganization of methodology but, in every case, such emphasis seems to be required by the very goals of teaching.

UNIVERSAL FUNCTIONS OF POLYGONAL NUMBERS*

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1. The four-square theorem of Lagrange. Among all positive integers an interesting subset, from the point of view of additive number theory, is the set of squares, that is, the integers 1, 4, 9, 16, 25, $\cdots$. Thus, for example, the integer 7, which is not in the subset of squares, is a sum of four integers in the subset, since $7=4+1+1+1$. No fewer than four summands would yield 7, and no other summands, apart from the order in which they are written in the equation, would yield 7. The integer 14, however, requires only three summands, and can not be obtained with exactly four summands. The integer 19 can be obtained with three summands or with four summands. Other positive integers require only two summands. The integer 7, for which three summands do not suffice, is not an isolated case. It was first proved by Gauss that three squares do not suffice for any of the positive integers in the infinite arithmetic progression $8n+7$. The general fact that four squares suffice for all positive integers was proved by Lagrange in 1770. This theorem, and many similar theorems, can be stated more simply if the integer 0 is permitted as a summand. Then the fundamental Lagrange theorem is that every positive integer is a sum of four squares.

This Lagrange theorem will be restated in terms that are more useful for purposes of generalization. The fact that a positive integer A is a sum of four squares is equivalent to the fact that the equation $A=x^2+y^2+z^2+w^2$ has a solution in integers which are greater than or equal to zero. It is then said that A is represented by the function $x^2+y^2+z^2+w^2$. A function which represents every positive integer is said to be *universal*. Hence the Lagrange theorem states that $x^2+y^2+z^2+w^2$ is *universal*.

2. Fermat-Cauchy theorem on polygonal numbers. The polygonal numbers of order 4 are precisely the squares, 0, 1, 4, 9, 16, $\cdots$. In general, if m is a positive integer, fixed but arbitrary, then the polygonal numbers of order $m+2$ are the integers which are the values of the expression $x+m(x^2-x)/2$ when x takes the values 0, 1, 2, 3, $\cdots$. Thus the polygonal numbers of order $m+2$ are the integers 0, 1, $m+2$, $3m+3$, $6m+4$, $10m+5$, $\cdots$.

The Lagrange theorem on representation of integers by sums of squares is a particular case of a general theorem on representation of integers by sums of polygonal numbers of order $m+2$. The first published proof of this general theorem was by Cauchy, but the theorem was first stated by Fermat. The Fermat-Cauchy theorem will be stated after it is illustrated further.

If $m=1$, the polygonal numbers of order 3 are the integers 0, 1, 3, 6, 10, 15, $\cdots$. They are the triangular numbers. The equations $14=10+3+1$, $7=6+1+0$, $19=15+3+1$ illustrate the general fact that every positive integer

* Delivered at a meeting of the Mathematical Association of America, Chicago, Illinois, September 1, 1941.

is a sum of three triangular numbers. It is not possible to use fewer than three triangular numbers to obtain 14.

If $m=6$ the polygonal numbers of order 8 are the integers 0, 1, 8, 21, 40, 65, $\cdots$. The equations $15=1\cdot 8+7\cdot 1$ and $19=2\cdot 8+3\cdot 1$ illustrate the general fact that every positive integer is a sum of 8 polygonal numbers of order 8, and that there are integers for which fewer polygonal numbers of order 8 would not suffice.

The expression $x+m(x^2-x)/2$ which defines the polygonal numbers of order $m+2$ is denoted by $p(x)$. Also p_1 denotes $p(x_1)$, p_2 denotes $p(x_2)$, and so on, but p_2 is not necessarily different from p_1. In these p_1, p_2, $\cdots$ the same m is used, although it is not explicitly indicated in the notation. The general Fermat-Cauchy theorem states that *every positive integer is a sum of $m+2$ polygonal numbers of order $m+2$, and that no fewer summands suffice for all positive integers.* These facts may be stated by saying that the function $p_1+p_2+\cdots+p_{m+2}$ is universal, and that no such function in fewer that $m+2$ variables is universal.

3. Dickson's generalization of the theorem of Lagrange. One generalization of the Lagrange theorem is illustrated by the function $x^2+y^2+z^2+2w^2$ and by the function $x^2+y^2+2z^2+3w^2$. Each of these functions is universal. In 1927 Dickson proved that there are exactly fifty-four such functions in four variables which are universal. If a function is obtained from one of these fifty-four functions by adjoining a term ct^2, in which c is a positive integer and t is a fifth variable, then the new function is trivially universal. Dickson proved that there are exactly six such functions in five variables, and no such functions in six or more variables, which are universal and which are such that no function obtained by deleting one or more terms is universal. Since no such function in three or fewer variables is universal, the problem of representation of integers by such functions $ax^2+by^2+cz^2+\cdots$ was completely solved.

4. Generalization of the Fermat-Cauchy theorem. In 1930 the author of this paper published the complete solution of the analogous generalization of the general Fermat-Cauchy theorem on polygonal numbers if $m\geqq 3$ and the sum of the coefficients is not greater than $m+2$. The universal functions $ap_1+bp_2+ep_3$ of triangular numbers were determined by Liouville in 1862. The general results for $m\geqq 3$ will be illustrated for $m=5$ and for $m=6$ before they are stated.

If $m=5$ the polygonal numbers of order 7 are the integers 0, 1, 7, 18, 34, 55, $\cdots$. The function $p_1+p_2+p_3+2p_4+2p_5$ is universal. This function will be designated by (1, 1, 1, 2, 2), since the coefficients are under investigation. That this function represents 13, for example, is indicated by the equation $13=1\cdot 7+1\cdot 1+1\cdot 1+2\cdot 1+2\cdot 1$. The other universal functions for $m=5$ are the Fermat function (1, 1, 1, 1, 1, 1, 1) and the functions (1, 1, 1, 1, 1, 2) and (1, 1, 1, 1, 3). That the function (1, 1, 1, 4) is not universal is illustrated by the fact that $17=2\cdot 7+3\cdot 1$ is the only representation of 17 as a sum of seven or fewer non-zero polygonal numbers of order 7.

If $m=6$ the polygonal numbers of order 8 are, as stated previously, 0, 1, 8, 21, 40, 65, $\cdots$. The universal functions are (1, 1, 1, 2, 3), (1, 1, 1, 1, 1, 3), (1, 1, 1, 1, 2, 2), (1, 1, 1, 1, 1, 1, 2), and the Fermat function (1, 1, 1, 1, 1, 1, 1, 1). It is clear that the representation of a positive integer A by the function (1, 1, 1, 2, 3) implies its representation by the function (1, 1, 1, 1, 1, 3), because the solution of the equation $A=p_1+p_2+p_3+2p_4+3p_5$ in integers $x_1, \cdots, x_5$ implies the solution of the equation $A=p_1+p_2+p_3+p_4+p_5+3p_6$ in integers $x_1, \cdots, x_6$. Hence the universality of the function (1, 1, 1, 2, 3) implies that of each of the other universal functions. It was not possible to use this fact in the proof, because the details of the proof were much more complicated for functions not having the first four coefficients each unity. The fact is mentioned, however, to emphasize the fact that the universality of each of the functions different from the Fermat function implies that of the Fermat function.

These results for $m=6$ constitute five theorems, including the Fermat theorem. Each of the other four theorems implies the Fermat theorem. The results for $m=5$ constitute four theorems, including the Fermat theorem. Each of the other three theorems implies the Fermat theorem. A contrast in the cases $m=5$ and $m=6$ is to be noted. The universal functions for $m=6$ can all be obtained from the one universal function having the fewest number of variables, that is from (1, 1, 1, 2, 3) by the method of partition illustrated above. However, for $m=5$ there are two universal functions, namely (1, 1, 1, 2, 2) and (1, 1, 1, 1, 3), each having the fewest number of variables, neither of which can be derived from the other by partition.

If m is a fixed but arbitrary integer which is greater than or equal to three, and n is the number of variables in a function, then the function is written $(a_1, a_2, \cdots, a_n)$. The coefficients $a_1, \cdots, a_n$ are integers each greater than or equal to unity, and have been arranged for convenience so that $a_1 \leqq \cdots \leqq a_n$. The sum $a_1+\cdots+a_k$ of the first k coefficients is denoted by w_k. The positive integer A is represented by the function $(a_1, \cdots, a_n)$ if and only if there exist integers $x_1, \cdots, x_n$, each of which is greater than or equal to zero, such that $A=a_1p_1+\cdots+a_np_n$. The function is universal if and only if it represents every positive integer.

The general results on universal functions of polygonal numbers, if $m \geqq 3$ and the sum of the coefficients is not greater than $m+2$, are easily stated, using these notations. If $w_n<m+2$ then there are no universal functions. If $w_n=m+2$ then the universal functions are the following functions: (1, 1, 1, 2); (1, 1, 1, 1, 3); $(1, 1, 1, 1, a_5, \cdots, a_n)$, with $a_5=1$ or 2, and with $a_k \leqq w_{k-1}-1$ $(5 \leqq k \leqq n)$; and $(1, 1, 1, 2, a_5, \cdots, a_n)$, with $a_k \leqq w_{k-1}-1$ $(5 \leqq k \leqq n)$.

The author of this paper has just published a determination, for $m \geqq 3$, of the minimum value N of n and a characterization of m such that there is a unique universal function having $n=N$. The results are that the minimum n is the integer N uniquely defined by the inequalities $2^{N-3}-1<m \leqq 2^{N-2}-1$, and that there is a unique universal function having $n=N$ if and only if $m=3, 4, 2^{N-2}-2, 2^{N-2}-1$. If $m \geqq 3$ then the integers $1, 1, 1, 2, 2^2, \cdots, 2^{N-4}, m+1-2^{N-3}$, after

they have been arranged in order of increasing magnitude, are the coefficients of a universal function having $n=N$. If $m=5$ then $N=5$ and the two universal functions having $n=N$ were noted to be (1, 1, 1, 2, 2) and (1, 1, 1, 1, 3). If m is odd and greater than 5 and not equal to $2^{N-2}-1$, then a set of integers which yield, after they have been arranged in order of increasing magnitude, a universal function having $n=N$ which is different from the function mentioned above, is the set $1, 1, 1, 2, 2^2, \cdots, 2^{N-5}, b, c$ with $b=(m+1-2^{N-3})/2$ and $c=m+1-2^{N-4}-b$. If m is even and greater than 4 and not equal to $2^{N-2}-2$ or 2^{N-3}, then this last set of integers with $b=(m+2-2^{N-3})/2$ and $c=m+1-2^{N-4}-b$ yields such a function. If $m=2^{N-3}$ then this last set of integers with $b=2$ and $c=m+1-2^{N-4}-b$ yields such a function.

The author has investigated universal functions of polygonal numbers of order $m+2$ in which $w_n \geqq m+3$. The methods of proof for the case $w_n \leqq m+2$ in general do not apply to the case $w_n \geqq m+3$. The proofs of the preceding facts for the case $w_n \leqq m+2$, and other references, will be found in the author's papers *A generalization of the Fermat theorem on polygonal numbers* and *A note on representation by polygonal numbers.**

DISCUSSIONS AND NOTES

EDITED BY R. J. WALKER, Cornell University, Ithaca, N. Y.

The department of Discussions and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

DIFFERENTIALS

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As easy and desirable as it is to get along without differentials, they seem to be with us to stay. The fact that differentials enable one to arrive at correct results when he thinks of dx and dy as "little pieces of x and y" makes him believe he is thinking correctly and he therefore will not give up his concept of differentials because some pedantic mathematician tells him he is just lucky.

Granted, then, that we are not going to dispense with differentials, can we not do something to help students understand what they are? The unsophisticated undergraduate who tries to believe everything he reads and his instructor tells him is hopelessly confused. One day he tries to believe (but does not succeed) that dy/dx is *not* dy divided by dx. The next day he may have momentary comfort when he learns that it is true after all that dy/dx is dy divided by dx and that $\lim_{\Delta x \to 0} \Delta y/\Delta x = dy/dx$. However, he is then told or, heaven forbid, sees "proved" that $dx=\Delta x$. He may admire the cleverness of dx for being able to

* Annals of Mathematics, (2), vol. 31, 1930, pp. 1–12, and a current issue of the Bulletin of the American Mathematical Society.

"remain constant on one side of an equation and (since $dx=\Delta x$) approach zero or the other," but it bothers him just the same.

In the usual way of representing a function (please do not fight with us about this functional notation) let x be the independent variable and $y=f(x)$ the dependent variable. Then let Δx be a number $\neq 0$ and such that $f(x+\Delta x)$ is defined but otherwise let Δx be arbitrary. Consider the number $f(x+\Delta x)-f(x)$ (if x insists on living up to its reputation as a variable, replace x by x_0), let

$$\Delta y,\ \Delta f,\ \Delta f(x)$$

be alternative notations for this number, and define each of the symbols

$$D_x y,\ D_x f,\ D_x f(x),\ y',\ f',\ f'(x)$$

to be the following limit, assumed to exist,

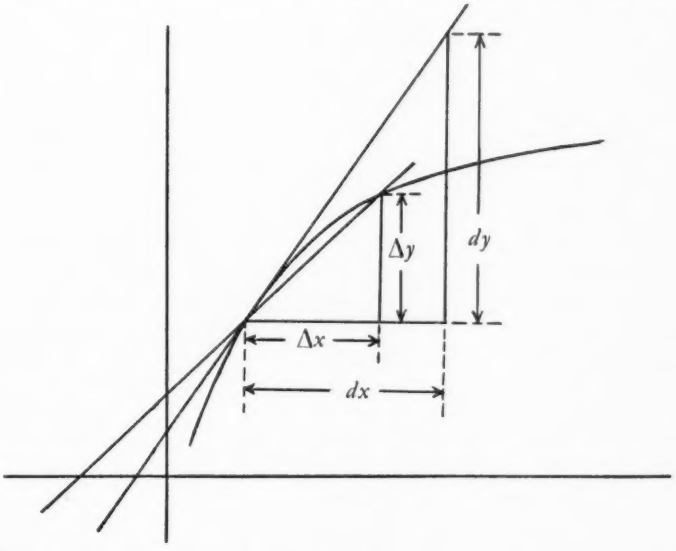
$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}.$$

Above all things do not define dy/dx to be this limit.

Now let dx be an absolutely arbitrary number and define

$$dy = f'(x)dx.$$

Clearly if $dx=0$ then $dy=0$, and if $dx\neq 0$ then $dy/dx=f'(x)$. Then from the very



beginning dy/dx, $dx\neq 0$, is dy divided by dx.

Since Δx and dx are quite arbitrary, there is no inherent reason why they

should be the same, and we therefore first take them different. Consequently, instead of the usual picture illustrating the relation of dy and Δy we have the geometric interpretation given in the figure. Since the slope of the chord approaches the slope of the tangent as $\Delta x \to 0$ we clearly have

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx},$$

and there is no worry about a strain on dx as $\Delta x \to 0$.

If now we wish to use differentials as approximations to increments, it should be clear to the student that dy is an approximation to Δy if we choose, as we may, $dx = \Delta x$ and "small." Also if it is thought that infinitesimals are desirable in this connection, then write

$$\lim_{dx=\Delta x \to 0} \frac{dy - \Delta y}{\Delta x} = \lim_{dx=\Delta x \to 0} \frac{dy}{dx} - \lim_{dx=\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{dx=\Delta x \to 0} f'(x) - f'(x) = f'(x) - f'(x) = 0,$$

and thus have that $dy - \Delta y$ is an infinitesimal of higher order than $\Delta x (= dx)$.

This definition of dx and dy amounts to what is essentially the Leibnitz definition of differentials; namely, dx is arbitrary and

$$dy = \lim_{t \to 0} \frac{f(x + tdx) - f(x)}{t}.$$

Differentials are, to be sure, not a great help when dealing with a function of one variable, but they become more important in the theory of functions of two or more variables. However, if differentials are introduced early in the course, their use will seem more natural to the student in later more complicated situations. It should, from the very beginning, be emphasized that a differential of an independent variable is itself independent, but a differential of the dependent variable is not independent but is determined.

In a course in mechanics when "virtual displacement", i.e., displacement along the tangent to a path, is introduced, the symbols δx and δy are generally used for the x- and y-components of this displacement. The reason why two symbols, δ's and d's, should be introduced when the algebraic manipulations are the same is never clear to the student. The student is right in this case; the δ's are only differentials and this symbol is therefore superfluous.

AN INDUCTIVE PROOF OF BUDAN'S THEOREM

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We shall be concerned with Budan's theorem in the following form:

If $f(x)$ is a polynomial of degree n with real coefficients, and $V(x)$ is the number of variations in sign of the sequence

$$f(x), f'(x), f''(x), \ldots, f^{(n)}(x),$$

then the number of real roots of $f(x)=0$ *on the interval* $a<x\leqq b$ *(a root of multiplicity* μ *being counted as* μ *roots) is* $V(a)-V(b)-2k$, *where* k *is a positive integer or zero.*

The proofs of this theorem which are given in the elementary textbooks usually involve a rather intricate examination of the behavior of the derivatives of $f(x)$. We present here an inductive proof involving a minimum of such detail.

Budan's theorem, as stated, is an immediate consequence of the following lemma.

LEMMA. *If* α *is a root of* $f(x)=0$ *of multiplicity** μ $(\mu\geqq 0)$, *then* $V(\alpha+0)=V(\alpha)$ *and* $V(\alpha-0)=V(\alpha)+\mu+2k(\alpha)$, *where* $k(\alpha)$ *is a positive integer or zero.*

Proof. We employ induction on the degree, n, of $f(x)$. If $n=1$, the lemma is obviously true. Suppose that it holds for all polynomials of degree at most $k-1$. Consider a polynomial $g(x)$ of degree k. Let $V_1(x)$ be the number of variations in sign of the sequence (1) with the first term suppressed and the f's replaced by g's. We divide the proof that $g(x)$ has the property of our lemma into two cases.

Case 1. $\mu>0$. The hypothesis of our induction applies to $g'(x)$, $V_1(x)$, and the root $x=\alpha$ of $g'(x)=0$ of multiplicity $\mu-1\geqq 0$. We obtain $V_1(\alpha+0)=V_1(\alpha)$ and $V_1(\alpha-0)=V_1(\alpha)+\mu-1+2k_1(\alpha)$, with $k_1(\alpha)$ a non-negative integer. But it is clear that $g(x)$ and $g'(x)$ have opposite signs just before and equal signs just after $x=\alpha$. Hence $V_1(\alpha-0)+1=V(\alpha-0)$ and $V_1(\alpha+0)=V(\alpha+0)$. But, since $g(\alpha)=0$, we have $V(\alpha)=V_1(\alpha)$. The desired result for $g(x)$ follows easily.

Case 2. $\mu=0$. Here $g(\alpha)\neq 0$, and the application of the hypothesis of our induction to $g'(x)$ and $V_1(x)$ yields $V_1(\alpha+0)=V_1(\alpha)$ and $V_1(\alpha-0)=V_1(\alpha)+\nu+2k_1(\alpha)$, with $k_1(\alpha)$ a non-negative integer and $\nu\geqq 0$ the multiplicity of the root $x=\alpha$ of $g'(x)=0$. If $\nu=0$, then $g'(\alpha)\neq 0$, and hence $V(\alpha-0)-V_1(\alpha-0)=V(\alpha)-V_1(\alpha)$ $=V(\alpha+0)-V_1(\alpha+0)$; from which the conclusion of our lemma for $g(x)$ follows easily. On the other hand, if $\nu>0$ we note that $g^{(\nu+1)}(\alpha)\neq 0$ and that $g'(\alpha)=g''(\alpha)=\cdots=g^{(\nu+1)}(\alpha)=0$, while $g'(x), g''(x), \cdots, g^{(\nu+1)}(x)$ alternate in sign just before $x=\alpha$ and all have the same sign just after $x=\alpha$. Hence $V(\alpha+0)-V_1(\alpha+0)=V(\alpha)-V_1(\alpha)$, which proves that $V(\alpha+0)=V(\alpha)$. If ν is even we see that $g'(x)$ and $g^{(\nu)}(x)$ have the same sign just before $x=\alpha$. Whether or not this is the sign of $g(\alpha)$, we obtain the conclusion of our lemma with $2k(\alpha)=\nu+2k_1(\alpha)$. If ν is odd and $g(x)$ and $g'(x)$ have the same sign just before $x=\alpha$, then $V(\alpha-0)=V_1(\alpha-0)$ and $V(\alpha)=1+V_1(\alpha)$, and it follows that $2k(\alpha)=\nu-1+2k_1(\alpha)$ is effective. Finally, if ν is odd and $g(x)$ and $g'(x)$ have opposite signs just before $x=\alpha$, then $V(\alpha-0)=V_1(\alpha-0)+1$ and $V(\alpha)=V_1(\alpha)$; and consequently $2k(\alpha)=\nu+1+2k_1(\alpha)$ is effective. This completes the induction and the proof of our lemma.

* For economy of statement we have designated a value $x=\alpha$ at which $f(x)\neq 0$ as "a root of $f(x)=0$ of multiplicity zero." The notation $V(\alpha+0)$ $[V(\alpha-0)]$ means the right-hand [left-hand] limit of $V(x)$ at $x=\alpha$.

RECENT PUBLICATIONS

EDITED BY VIRGIL SNYDER, Cornell University

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

A New System of Reckoning which Turns at 8. By Emanuel Swedberg. Translated from a photostat copy of the original Swedish MS. by Alfred Acton. Philadelphia, Pa., Swedenborg Scientific Association, 1941. 34 pages. 60 cents.

This translation from an original manuscript written in 1718 will be of interest to two groups. Collectors of Swedenborgia will welcome this addition for its own sake while mathematicians will be interested from a theoretical and practical point of view. Students of Swedenborg will read the translator's preface with greater attention than the actual translation because of the generous quotations from two other papers of Swedenborg, setting the historical scene for this particular work. They will notice that the earlier spelling Swedberg is used since the change in spelling did not occur until after this paper was written.

Mathematicians will concentrate on the paper itself which contains a short preface by the author followed by a detailed discussion of the new system consisting of eight characters made up of letters. Directions for reading and writing the corresponding numbers are carefully given. Addition and multiplication tables are compiled with diagrams for Napier's sticks using the octonary system. There follow sections on applications to monetary problems and problems on weights and measures. Although the arguments are quite convincing, they would be more so if the translator had not confessed in his preface that Swedberg himself advocated in 1719 the use of the decimal system. It would have been apropos here to state that the decimal system for all weights and measures was introduced into Sweden in 1879 and became obligatory in 1889.

The translator has been very diligent in discovering errors in the manuscript. Unfortunately, there are several mistakes in the translation, some of them typographical. For the sake of those students of Swedenborg who may not be trained mathematicians, we list the mistakes noted in reviewing the paper:

p. 18: in the *Addition Table*, the end of the second line of the second column should read *ll* [*lyl*] instead of [*ll*] *lyl*.

p. 19: the last two problems in multiplication should be arranged as follows:

$$\begin{array}{r} nm \\ sn \\ \hline lsm \\ vo \\ \hline losm \end{array} \qquad \begin{array}{r} mv \\ ls \\ \hline llf \\ mv \\ \hline fof \end{array}$$

p. 22: in the example at bottom of page. The last two lines should have *l* and 1 under *Riksd.* instead of under *Mark.*

p. 28: line 2 from top. The expressions *l*;*m*,*nn*;*nm*; should be (*l*);(*m*,*nn*);(*nm*). Also in line 6 from top the expressions *l*;*nn*;*mn*; should be (*l*);(*nn*);(*mn*).

p. 28: footnote marked †. The computation should be arranged as follows:

Riksd.

$$\begin{array}{r} n\ n \\ m\ n \\ \hline l\ s\ l \\ t\ n \\ \hline f\ o\ f \end{array}$$

p. 29: fourth line above footnotes: $1\frac{85}{84}$ should be $1\frac{89}{84}$.

p. 30: fourth line above footnotes: 3065 should be 3675.

p. 30: footnote marked ‡, next to last line should have the number 1 under the column headed 1/2 *quart.*

p. 33: line 10 should read: the cube root of l_0 is s.

HARRIET F. MONTAGUE

CLUBS AND ALLIED ACTIVITIES

EDITED BY E. H. C. HILDEBRANDT AND J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest, to J. S. Frame, Brown University, Providence, R. I.

BIBLIOGRAPHY ON METHODS OF APPORTIONMENT IN CONGRESS*

On November 15, 1941, President Roosevelt approved the automatic apportionment bill (H.R.2665, now Public Law 291) which replaces the method of major fractions (devised by Professor W. F. Willcox of Cornell in 1910) by the method of equal proportions (devised by Professor E. V. Huntington of Harvard in 1921). The twenty-year controversy thus brought to an end affords an instructive example of the application of the scientific method to a political problem. The following bibliography may therefore be a timely convenience for students of this problem.

For the most comprehensive list of numerical examples illustrating all known methods, see Huntington's *A Survey of Methods of Apportionment in Congress*, Senate Document No. 304, 76th Congress, 3rd Session, 1940 (Government Printing Office, 41 pages, $0.10). On legal aspects, see Z. Chafee, Jr.'s article on *Congressional reapportionment* in the Harvard Law Review, vol. 12, pp. 1015–1047,

* Compiled from data supplied at our request by E. V. Huntington.

1929, and his forthcoming paper on *Reapportioning the house of representatives under the 1940 census* in the Proceedings of the Massachusetts Historical Society, vol. 66, 1942. See also L. F. Schmeckebier's book on *Congressional Apportionment*, Brookings Institution, Washington, 1941. On the theoretical side, the fundamental paper is Huntington's *The apportionment of representatives in congress*, Transactions of the American Mathematical Society, vol. 30, pp. 85–110, 1928. The text of the main part of the bill is given in the Congressional Record for October 20, 1941, p. 8261.

Hearings before the House Committee on the Census have been held as follows, but must be read with caution: December 1920 to January 1921 (H.R. 14498, etc.); January to February, 1927 (H.R. 13471); February 1928 (H.R. 130); February 1931 (H.R. 15983, etc.); February to March, 1940. Also, before the Senate Committee on Commerce, February to March, 1941 (H.R. 2665).

For the report of the census advisory committee, see the *Quarterly Publication of the American Statistical Association*, December, 1921, pp. 1004–1013; or the *Congressional Record* for April 7, 1926, pp. 6840–6842; or the 1927 *Hearings*, pp. 53–58; or the 1928 *Hearings*, pp. 80–85. For the report of the National Academy of Sciences, see the *Annual Report of the Academy* for 1928–1929, pp. 20–23; or the *Congressional Record* for March 2, 1929, p. 5059; or the 1940 *Hearings*, pp. 70–71.

For the origin of the method of equal proportions, see Huntington, *A New Method of Apportionment of Representatives*, Quart. Pub. Amer. Statistical Assoc., September 1921, pp. 859–870; or a brief abstract in the *Proceedings of the National Academy of Sciences*, vol. 7, pp. 123–127, April 1921. On the origin of the method of major fractions, see Willcox, *House Report No. 12*, 62nd Congress, 1st Session, April 15, 1911; or his presidential address as president of the American Economic Association in the *American Economic Review*, vol. 6, no. 1, supplement, pp. 1–16, March, 1916, or in the 1927 *Hearings*, pp. 79–86. See also F. W. Owens' paper *On the apportionment of representatives*, Quart. Pub. Amer. Statistical Assoc., December, 1921, pp. 958–968. On Dr. Hill's method of alternate ratios, see J. A. Hill, *House Report No. 12, loc. cit.*, 1911, and his statement accepting the method of equal proportions, on p. 16 of the 1927 *Hearings*.

An interchange of papers by Huntington and Willcox may be found in *Science*: Huntington, vol. 67, pp. 509–510, May 18, 1928; vol. 68, pp. 579–582, December 14, 1928; Willcox, vol. 69, pp. 163–165, February 8, 1929; Huntington, vol. 69, p. 272, March 8, 1929; Willcox, vol. 69, pp. 357–358, March 29, 1929; Huntington, vol. 69, pp. 471–473, May 3, 1929. See also Huntington, *American Political Science Review*, vol. 25, pp. 961–965, November, 1931. Also C. L. Dedrick's paper in the 1941 *Hearings*, pp. 42–44, and a short statement by Huntington, printed in the Appendix of the *Congressional Record* for April 28, 1941, page A 2053.

The controversial aspect of the situation has culminated in a tripartite article published in *Sociometry*, vol. 4, August, 1941: *The role of mathematics in*

congressional apportionment, pp. 278–282, by Huntington; a *Reply*, pp. 283–298, by Willcox; and a *Rejoinder*, pp. 299–301, by Huntington.

I DOUBT IT—A MATHEMATICAL CARD GAME

IRENE PRICE, Oshkosh State Teachers College

The cards for this game consist of a deck of 50 or more cards which may be made by cutting cardboard into small rectangles (or some other geometric figure) and placing a number from 0 to 7 on each one. The numbers are not written in the usual manner but in some symbolic way so that the players must do a little work before knowing the number. *Zero* may be written in the following ways: $1+\cos 180°$, $\log 1$, $\cos \pi/2$, $0-0$, $6°-1$, etc. *One* may be written i^4, $\cos 0°$, $\log_e e$, $\tan 225°$, etc. For *two* write $d(2x)/dx$, $4^{1/2}$, $\cos^2 45°$, ${}_2C_1$, $\sqrt{12}/\sqrt{3}$, $\sec \pi/3$, $2\tan^2 225°$. For *three* write $\sqrt[4]{81}$, $3\sin 90°$, $\sqrt{7+\sqrt{4}}$, $3(5)°$, $\cot^2 30°$, $6/\sqrt[3]{8}$, etc.

Choose the numbers in such a way that they will be within the ability of the group; that is, if students have not studied trigonometry, then omit those forms which contain trigonometric functions; if the students know calculus, add cards containing integrals and derivatives.

This game may be played by any number of players from 2 to 8. The cards are distributed to the players, one at a time, until all cards are given out. The dealer starts the game by placing a card face down on the table and calling it *zero*. The player to his left places a card face down on the table in front of him and calls "one"; the next player does likewise and says "two," and so on around the table. If at any time a person doubts that the player put down the number he declares, the challenger says, "I doubt it" at which the player exposes the card played. If it was the card declared, the challenger takes all of the cards the player has on the table in front of him; if it was not the card declared the player must take all of the cards the challenger has before him. The object of the game is to see who can get out of cards first. When all cards in the hand have been played, the player must pick up all the cards he has placed upon the table and play them again. After counting to seven the next player may lay down any number he wishes but the next players to his left must then lay down the next consecutive numbers until 7 is reached.

BOOKS FOR CLUBS

(*Continued from December 1938*)

46. *What is Mathematics? An Elementary Approach to Ideas and Methods*, by R. Courant and H. Robbins. New York, Oxford, 1941. 521 pages. $3.75. Contains a wealth of material for new topics for discussion at club meetings. A few samples are: repeated reflections, drawing with mechanical instruments, Schwarz's triangle problem, Steiner's problem, Appollonius' problem, an instrument for doubling the cube.

47. *Fundamental Mathematics*, by D. Harkin. New York, Prentice-Hall, 1941.

434 pages. $3.00. Secondary school as well as college clubs will find many worthwhile subjects for consideration, such as: development of the number concept, the decimal system, finite modular arithmetic, stories about fractions, symbolism, continued fractions, tesselation, stellated polyhedra, and many others.

48. *Fundamentals of Mathematics*, by M. Richardson. New York, Macmillan, 1941. 525 pages. $3.25. One chapter entitled "Impossibilities and unsolved problems" discusses a number of popular subjects. Other interesting topics as well as references for further study can be found.

49. *Mathematics—Its Magic and Mastery*, by A. Bakst. New York, Van Nostrand, 1941. 790 pages. $3.95. Program committees interested in attractive topics for club meetings will find interest aroused in such chapter headings as: how to multiply and like it, Napier's escape from drudgery, every number has its fingerprint, algebra hits the jackpot, the perils of flatland, railroading among the stars. The chapter on ballistics contains an unusually simple introduction to the subject.

KAPPA MU EPSILON

FIFTH BIENNIAL CONVENTION

The fifth biennial convention of *Kappa Mu Epsilon* was held at Central Missouri State Teachers College, Warrensburg, Missouri, on April 18 and 19, 1941, with the *Missouri Beta* chapter as host. Twenty-three chapters were represented by one hundred and fifty-nine delegates and visiting members. The convention opened with a general assembly on Friday evening which included an address by Dr. W. C. Morris of Warrensburg entitled *Playing a little with electricity*, and a demonstration of a reed organ converted into an electronic organ by George W. Wood, a student at Warrensburg. The educational program on Saturday morning consisted of two student papers, *Projective measurements* by William Wallis of *Texas Alpha* and *Some interesting facts about the cycloid* by Jerry Falvey of *Louisiana Alpha*, and discussions by Professor O. J. Peterson of *Kansas Beta* and Professor E. R. Sleight of *Michigan Alpha* on *ABCD* and *Early English arithmetics*, respectively. Professor C. V. Newsom, national president, presided at all meetings. Professor Emmett Ellis of Warrensburg was the convention chairman.

A number of revisions of the constitution were made at the business meeting on Saturday afternoon. The fraternity voted to publish an official journal at least once a year. Sectional meetings were encouraged but mathematical contests were not favored. The convention concluded with a banquet on Saturday evening at which Albro F. Stepp, student at *Missouri Beta*, served as toastmaster.

The following were elected to serve as the officers for the next biennium:

President Pythagoras, Professor O. J. Peterson, *Kansas Beta*, State Teachers College, Emporia, Kansas

Vice-President Euclid, Professor E. D. Mouzon, *Texas Beta*, Southern Methodist University, Dallas, Texas

Secretary Diophantus, Professor E. Marie Hove, *Nebraska Alpha*, State Teachers College, Wayne, Nebraska

Treasurer Newton, Professor H. Van Engen, *Iowa Alpha*, State Teachers College, Cedar Falls, Iowa

Historian Hypatia, Miss Orpha Ann Culmer, *Alabama Beta*, State Teachers College, Florence, Alabama

Past President Zeno, Professor C. V. Newsom, *New Mexico Alpha*, University of New Mexico, Albuquerque, New Mexico.

UNDERGRADUATE PUBLICATIONS

The Pentagon, official journal for *Kappa Mu Epsilon*, national honorary fraternity in mathematics, made its initial appearance in the fall of 1941. Edited by Professor C. V. Newsom of the University of New Mexico, its sixty-eight pages contain articles on *Mathematics and national defense* by W. L. Hart, *Dyadic arithmetic* by H. D. Larsen, and *Robert Record's Whetstone of Witte* by E. R. Sleight as well as a section devoted to *The mathematical scrapbook* and a summary of activities of the twenty-six chapters. Professor E. R. Sleight of Albion College, Albion, Michigan, will edit all student papers submitted for publication. Professor E. A. Hazlewood, Texas Technological College, Lubbock, Texas, is making a collection of items for *The mathematical scrapbook*. News notes are submitted to Miss Orpha Ann Culmer, Alabama State Teachers College, Florence, Alabama or to Miss E. Marie Hove, Nebraska State Teachers College, Wayne, Nebraska. Clubs or individuals not members of the national fraternity may subscribe for the magazine at thirty cents a copy or one dollar for two years. Subscriptions and business communications should be sent to C. B. Barker, University of New Mexico, Albuquerque, New Mexico.

CLUB REPORTS, 1940–41

Pi Mu Epsilon, University of Arizona

The 40th chapter of *Pi Mu Epsilon* was installed on April 7, 1941 at the University of Arizona at Tucson. The installation ceremony was conducted by W. E. Milne, Director-General of *Pi Mu Epsilon*, with the assistance of Professor James B. Shaw. Thirty-one of the thirty-three charter members were initiated into membership during the ceremony. At a banquet following the installation, Professor Milne gave an address on *Primitive number systems*.

Oberlin Mathematics Club

References used in preparing papers for club meetings are always welcomed by this department. The report of the Oberlin Mathematics Club lists a number which were used. John Hammerle spoke on *Boolean algebra* and concluded his talk with applications given in the paper *Relations and reason* by W. V. Quine in the *Technology Review*, vol. 41, 1939, p. 299. Allen Strehler gave a demonstration of soap films with wire loops based on *Soap film experiments with minimal surfaces* by R. Courant, this MONTHLY, vol. 47, pp. 167–174. Roselyn Siegel found the article on map making in the *Encyclopedia Britannica*, 14th Edition, an excellent guide. Other speakers were Dr. R. W. Wagner on *Topology*, Bolton Strauch on *Applications of life and interest tables to the problems of life insurance and annuities*, Miss Orpah Clark on *The regular solids and related polyhedrons and crystals*, and William Hosier on *Continued fractions* (see Hall and Knight's *Higher Algebra*). In addition short biographical sketches of mathematicians chosen from Bell's *Men of Mathematics* were given at each meeting and a student contest held was based on twenty-five trick questions and puzzles and modeled on the plan of the radio program *Information Please*.

Regis College Mathematical Club

A typical meeting includes talks and demonstrations on popular mathematical subjects and some time devoted to recreational mathematics. Mary Kelly gave a demonstration of the ruler and compasses construction of a Clifford chain and Margaret McCarthy at a later meeting explained the analytical approach to the Clifford chain problem. Sister Thomas á Kempis gave a digest of *The walking polygot* which appeared in *Scripta Mathematica*, December 1939. Teresa Launie discussed *Mathematics and defense* in which she stressed the importance of training in mathematics and discussed applications in ballistics and used for reference the field and coast artillery manuals published by the U. S. Government Printing Office for the Army and Navy. Evelyn Carrellas described *The Snellius approximation and its construction and application to the three famous problems of antiquity*, from Dantzig's *Number, the Language of Science. The fourth dimension* was discussed by Eleanor Fleming, *Primitive counting* by Marjorie Sullivan, *Evariste Galois* by Gertrude McDonald, *The life of Pascal* by Cecelia Reininger, and *The trisection problem* by Helen Cleary.

Harold Young, superintendent of personnel of the Employers Insurance Group of Boston, spoke on *Insurance as a future for mathematics majors.* The club also participated in the activities of the Greater Boston Mathematics Clubs Association, acting as host for the April 1941 meeting. Entertainments at meetings were based on Ball's *Mathematical Recreations and Essays*, Jones' *Mathematical Wrinkles*, and Steinhaus' *Mathematical Snapshots*. Officers were: President, Irene Thomas; Vice-President, Cornelia Dinneen; Secretary, Eleanor Fleming; Treasurer, Marjorie Sullivan; Faculty Adviser, Sister Leonarda.

Mathematics-Physics Club, Haverford College

Topics discussed at meetings included: *The problem of apportionment of representatives* by R. B. Dickson, *Problems in television* by T. H. Chambers, *Fermat's last theorem* by R. Strohl, *Some figures in topology* by Dr. E. E. Betz, *Development of Wilson cloud chambers* by K. A. Wright, and *Linear Diophantine equations* by P. R. O'Connor. Guest speakers were Professor Arnold Dresden of Swarthmore College who gave a talk entitled *Intuition in mathematics*, and Dr. T. F. Anderson, RCA research fellow, who gave an illustrated lecture at a joint meeting of the Mathematics-Physics, Biology, Engineering, and Chemistry Clubs of the college on *The electron microscope*. Winners of the freshman prize examination were R. Day and E. C. Alvord, Jr. Officers were: President, G. R. Strohl, Jr.; Secretary, W. Franzen.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR. AND H. S. M. COXETER

ELEMENTARY PROBLEMS

Send communications concerning Elementary Problems and Solutions to H. S. M. Coxeter, 69 Chaplin Crescent, Toronto, Canada.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 506. *Proposed by R. V. Heath, Wall St., New York.*

Show that, for every positive integer n, the last $n+9$ digits of

$$90625^{2^n}$$

form an automorphic number. [See 1941, 407.]

E 507. *Proposed by V. Thébault, San Sebastián, Spain.*

In an orthocentric tetrahedron with orthocenter H and circumcenter O, show that the radical planes of the circumsphere with the respective spheres whose diameters are the four medians, meet the Euler lines of the corresponding faces in four points lying in a plane perpendicular to OH. (Cf. E 467. The *medians* of a tetrahedron join its vertices to the centroids of the opposite faces.)

E 508. *Proposed by R. K. Allen, Montpelier, Vermont.*

How many bridge hands are there where all thirteen tricks can be taken at no trump regardless of the distribution of the cards? It is assumed that declarer

will always play his highest cards and not intentionally lose any tricks. (Cf. E 448.)

E 509. *Proposed by the late J. E. Trevor, Cornell University.*

An architect is designing a house for his client's seventy-five foot lot. The "square bedroom" is to have a square floor, and it will contain an ordinary double bed and other furniture. The owner specifies that the music room shall be two feet longer than it is wide, and that its floor-area in square feet shall be three times that of the square bedroom. It is also specified that the widths of the two rooms shall be integer numbers of feet. Find these widths.

E 510. *Proposed by S. H. Gould, University of Toronto.*

Given $a_{p,q}=a_{p,q-1}+qa_{p-1,q}$, $a_{p,1}=a_{1,q}=1$, $(p, q=1, 2, 3, \cdots)$, prove

$$(p-1)a_{p,q}=\sum_{k=1}^{p-1}a_{p-k,q}\left\{(q+1)^{k+1}-\binom{p+q}{k+1}\right\}.$$

SOLUTIONS

E 464 [1941, 210]. *Proposed by Emma Lehmer, Berkeley, Calif.*

Prove that, for any prime $p>3$,

$$\binom{kp^{\alpha}}{np^{\beta}}\equiv\binom{kp^{\alpha-\gamma}}{np^{\beta-\gamma}}\pmod{p^{\alpha-\gamma+2}}.$$

Solution by the Proposer.

This result follows from repeated applications of the corresponding theorem in which $\gamma=1$, namely,

$$\binom{kp^{\alpha}}{np^{\beta}}\equiv\binom{kp^{\alpha-1}}{np^{\beta-1}}\pmod{p^{\alpha+2}}.$$

We first consider those factors of the binomial coefficient

$$\binom{kp^{\alpha}}{np^{\beta}}=\frac{kp^{\alpha}(kp^{\alpha}-1)(kp^{\alpha}-2)\cdots(kp^{\alpha}-np^{\beta}+1)}{1\cdot 2\cdots(np^{\beta}-1)np^{\beta}}$$

which are multiples of p; and after dividing out p from each factor of the numerator and denominator, we obtain

$$\frac{kp^{\alpha-1}(kp^{\alpha-1}-1)\cdots(kp^{\alpha-1}-np^{\beta-1}+1)}{1\cdot 2\cdots(np^{\beta-1}-1)np^{\beta-1}}=\binom{kp^{\alpha-1}}{np^{\beta-1}}.$$

The remaining factors can be grouped into batches of $p-1$ ratios each of factors lying between any two multiples of p, as follows:

$$\frac{\{kp^\alpha-(tp+1)\}\{kp^\alpha-(tp+2)\}\cdots\{kp^\alpha-(tp+p-1)\}}{(tp+1)(tp+2)\cdots(tp+p-1)}$$

$$=1-kp^\alpha\sum_{r=1}^{p-1}\frac{1}{tp+r}+k^2p^{2\alpha}\sum_{r\neq s=1}^{p-1}\frac{1}{(tp+r)(tp+s)}-+\cdots\equiv 1 \pmod{p^{\alpha+2}}.$$

For, it is well known that any elementary symmetric function of the first $p-1$ integers (or of any $p-1$ integers lying between two multiples of p) is divisible by p; moreover, the expression

$$\sum_{r=1}^{p-1}\frac{1}{tp+r}\equiv\sum_{r=1}^{p-1}\frac{1}{r}-pt\sum_{r=1}^{p-1}\frac{1}{r^2}$$

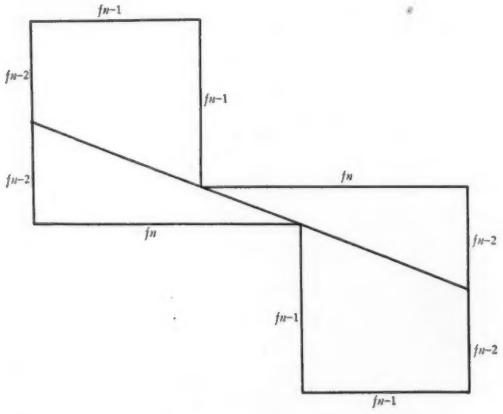
is divisible by p^2, since $\sum(1/r)$ is divisible by p^2 (for $p>3$) by Wolstenholme's theorem [1941, 269], and $\sum(1/r^2)$ is divisible by p.

E 468 [1941, 266]. *Proposed by W. R. Ransom, Tufts College.*

The Fibonacci numbers, defined by $f_1=f_2=1$, $f_{j+1}=f_{j-1}+f_j$, are known to yield a puzzle in which a square of side f_n is cut into four pieces which can apparently be rearranged to form a rectangle $f_{n-1}\times f_{n+1}$. Show that the same four pieces can be rearranged to form a figure which appears to consist of two rectangles $f_{n-1}\times 2f_{n-2}$ connected by a rectangle $f_{n-4}\times f_{n-2}$, the error being again one unit of area.

Solution by H. W. Eves, Pittsburgh, Pa.

We divide the given square into four pieces and rearrange these as in W. W. Rouse Ball's *Mathematical Recreations and Essays*, London, 1940, p. 85. To



show that the error in area is one unit, we observe that

$$f_{n+1}f_{n-1}=f_n^2+(-1)^n. \tag{1}$$

This is readily proved by induction as follows. Assume that (1) holds for $n=k$; then we have

$$\begin{aligned} f_{k+2}f_k &= (f_{k+1}+f_k)f_k = f_{k+1}f_k + f_k^2 \\ &= f_{k+1}f_k + f_{k+1}f_{k-1} - (-1)^k \\ &= f_{k+1}(f_k + f_{k-1}) - (-1)^k \\ &= f_{k+1}^2 + (-1)^{k+1}. \end{aligned}$$

The second rearrangement, depicted above, is possible since

$$f_n - f_{n-1} = f_{n-2}$$

and

$$2f_{n-2} - f_{n-1} = f_{n-2} - (f_{n-1} - f_{n-2}) = f_{n-2} - f_{n-3} = f_{n-4}.$$

To show that the error in area is again one unit, we notice that

$$\begin{aligned} 4f_{n-1}f_{n-2} + f_{n-2}f_{n-4} &= 4f_{n-1}f_{n-2} + f_{n-2}(2f_{n-2} - f_{n-1}) \\ &= f_{n-2}(4f_{n-1} + 2f_{n-2} - f_{n-1}) \\ &= f_{n-2}(3f_{n-1} + 2f_{n-2}) \\ &= f_{n-2}(f_n + f_{n-2} + 2f_{n-1}) \\ &= f_nf_{n-2} + f_{n-2}^2 + 2f_{n-1}f_{n-2} \\ &= f_{n-1}^2 - (-1)^n + f_{n-2}^2 + 2f_{n-1}f_{n-2} \\ &= (f_{n-1} + f_{n-2})^2 - (-1)^n \\ &= f_n^2 - (-1)^n. \end{aligned}$$

Also solved by the proposer.

E 469 [1941, 266]. *Proposed by Virgil Claudian, Bucharest, Roumania.*

Show that the exradii and circumradius of a triangle satisfy the identity

$$\sum \frac{a^2(b^2-c^2)}{r_a(r_b^2-r_c^2)} = 4R.$$

Solution by C. W. Trigg, Los Angeles City College.

From the well known relations

$$r = \frac{\Delta}{s}, \quad r_a = \frac{\Delta}{s-a}, \quad \frac{1}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}, \quad \text{and} \quad \Delta^2 = rr_ar_br_c,$$

it may be shown that

$$(r_b + r_c)(r_a - r) = a^2 \quad \text{and} \quad (r_b - r_c)(r_a + r) = b^2 - c^2.$$

It follows that

$$\sum \frac{a^2(b^2-c^2)}{r_a(r_b^2-r_c^2)} = \sum \frac{r_a^2-r^2}{r_a} = \sum r_a - r^2 \sum \frac{1}{r_a} = r_a + r_b + r_c - r = 4R.$$

(See Altshiller-Court, *College Geometry*, p. 73.)

Also solved by F. A. Alfieri, H. W. Bailey, W. E. Buker, H. W. Eves, Albert Furman, Edward Smith, E. P. Starke, and P. D. Thomas.

E 470 [1941, 266]. *Proposed by W. E. Buker, Pittsburgh Public Schools.*

Circle I has its center on another circle J. They intersect at A and C. From any point B on J, draw BC, intersecting I again at D. Prove that $BD=BA$.

Solution by D. H. Browne, Buffalo, N. Y.

Let O be the center of I. Since the arcs AO and CO are equal, BO bisects $\angle ABC$ (internally or externally). Hence D is the image of A by reflection in BO, and $BD=BA$.

Also solved by F. A. Alfieri, W. B. Clarke, H. A. DoBell, William Douglas, H. W. Eves, Albert Furman, F. W. Morgan, E. P. Starke, P. D. Thomas, C. W. Trigg, and the proposer.

E 471 [1941, 337]. *Proposed by L. G. Johnson, Ann Arbor, Michigan.*

A watch attached to a chain is swung around in a circle with the same angular velocity as that of its second hand. Show that the path traced by the tip of the second hand is a limaçon or a circle according as the sense of motion is clockwise or counterclockwise.

Solution by E. P. Starke, Rutgers University.

Let T be the tip of the second hand, and P its axis. Choose the fixed point of the chain (around which the watch is swung) as the origin, and let the positive x-axis be the direction in which PT points directly away from O. Let the lengths OP and PT be designated by a and b, respectively. Then at any instant the coördinates of P are $(a\cos\theta, a\sin\theta)$, and those of T are

$$(1) \qquad x = a\cos\theta + b\cos(\theta+\phi), \quad y = a\sin\theta + b\sin(\theta+\phi),$$

where θ is the angle POX and ϕ is the angle which PT makes with OP produced. The two cases of the problem are given by $\phi=\theta$ and $\phi=-\theta$. For $\phi=\theta$, (1) becomes

$$x = a\cos\theta + b\cos 2\theta, \qquad y = a\sin\theta + b\sin 2\theta,$$

or

$$x + b = (a + 2b\cos\theta)\cos\theta, \qquad y = (a + 2b\cos\theta)\sin\theta,$$

which is a limaçon, having the polar equation $r=a+2b\cos\theta$ if the pole is taken

at $(-b, 0)$. On the other hand, for $\phi = -\theta$, (1) becomes the circle

$$x = a \cos\theta + b, \qquad y = a \sin\theta,$$

with center $(b, 0)$ and radius a.

E 451 [1941, 65]. *Proposed by W. E. Buker, Pittsburgh Public Schools.*

There are three containers, having capacities of a, b, c quarts, where $a > b > c$ (positive integers). With the largest container full and the others empty, it is desired to divide the liquid into two equal portions, using these containers and no others. For what values of a, b, c is a solution possible?

Partial Solution by D. H. Browne, Buffalo, N. Y.

It is implied, of course, that a is even. The successive operations consist, essentially, in increasing or decreasing the amount of liquid in the largest container by b or c quarts. There is no loss of generality in assuming b and c to be relatively prime; for, any common divisor d must divide $\frac{1}{2}a$ also, and we merely have the problem for $a/d, b/d, c/d$, in terms of a new unit equal to d quarts. It is found that solutions are always possible when

$$b + c - 2 \leqq a \leqq 2(b + c).$$

(See Uspensky and Heaslet, *Elementary Number Theory*, p. 184.) The latter inequality is clearly necessary in order to be able to reduce the liquid in the largest container by as much as $\frac{1}{2}a$ quarts. There are certain "adventitious" solutions with $a < b+c-2$; e.g. when $a=22$, $b=16$, $c=9$, or $a=40$, $b=27$, $c=17$. In these instances the successive amounts in the three containers are as follows:

22	0	0	40	0	0
13	0	9	13	27	0
13	9	0	13	10	17
4	9	9	30	10	0
4	16	2	30	0	10
20	0	2	3	27	10
20	2	0	3	20	17
11	2	9	20	20	0

Also solved to this extent by the proposer.

Editorial Note. The actual solution for given values of a, b, c can be written concisely by giving the successive amounts in the two smaller containers together. This amount is initially 0 and finally $\frac{1}{2}a$. Thus, in the above instances we have the sequences

$$0, 9, 18, 2, 11; \quad 0, 27, 10, 37, 20.$$

Adopting the notation of H. D. Grossman's *Generalization of the water-fetching puzzle* [1940, 374], we see that such a sequence occurs as part of the cycle of s's or σ's, defined as follows:

$$s_0 = 0, \quad s_1 = b, \quad s_{i+1} = s_i + b \text{ or } s_i - c \text{ according as } s_i < c \text{ or } s_i \geqq c.$$

The definition for σ_k is the same with b and c interchanged. The complete cycle of σ's is just the cycle of s's written backwards. If b and c are relatively prime, there are $b+c$ distinct numbers in the cycle, namely $s_0, s_1, \cdots, s_{b+c-1}$, and these are a permutation of $0, 1, \cdots, b+c-1$. Thus we can solve the problem if and only if, in running round the cycle, either forwards or backwards, from 0 to $\frac{1}{2}a$, we do not pass any number greater than a.

In the case when $a=22$, $b=16$, $c=9$, the cycle of s's (with the relevant part emphasized) is as follows:

0, 16, 7, 23, 14, 5, 21, 12, 3, 19, 10, 1, 17, 8, 24, 15, 6, 22, 13, 4, 20, **11**, **2**, **18**, **9**.

Here we could not have $a=20$ instead of 22, since it is impossible to go from 0 to 10 without passing 21 or 22.

Since the cycle contains no number greater than $b+c-1$, the forward and backward routes from 0 to $\frac{1}{2}a$ are *both* available whenever $a \geqq b+c-1$ (giving different solutions). When $a=b+c-2$, one of these routes is blocked by the number $b+c-1$, but the other is still open. The following theorem provides a sufficient condition for the existence of solutions with $a<b+c-2$:

If $c \equiv \pm r \pmod{b-c}$, *where* $0<r<\frac{1}{2}(b-c)-1$, *there are solutions for* $a=2(c+nr)$ *with* $n=0, 1, 2, \cdots$.

Proof. If $c=m(b-c)-r$, we have

$$s_{2h-1} = hb - (h-1)c \quad \text{and} \quad s_{2h} = hb - hc$$

for $h \leqq m$, but $s_{2m+1}=mb-(m+1)c=r$. The greatest of the first $2m+1$ s's is $s_{2m-1}=2c+r$. The next batch of s's is given by

$$s_{2m+1+k} = s_k + r \tag{1}$$

for $k \leqq 2m+1$, and the greatest of these is $s_{2m-1}+r=2c+2r$. The sequence continues to satisfy (1) so long as the "maxima"

$$s_{n(2m+1)-2} = 2c + nr$$

remain less than $b+c$. Hence we can proceed from $s_0=0$ to

$$s_{n(2m+1)-1} = c + nr = \tfrac{1}{2}a$$

without passing any number greater than $2c+nr=a-nr$.

If $c=m(b-c)+r$, we may suppose $m>0$ (since the solution is trivial when $\frac{1}{2}a$ is a multiple of c). Thus we have

$$\sigma_{2h-1} = hc - (h-1)b \quad \text{and} \quad \sigma_{2h} = (h+1)c - (h-1)b$$

for $h \leqq m+1$, including $\sigma_{2m+1}=(m+1)c-mb=r$ and $\sigma_{2m+2}=c+r$. The greatest of the first $2m+1$ σ's is $\sigma_2=2c$. Thereafter, we have $\sigma_{2m+1+k}=\sigma_k+r$ so long as the maxima

$$\sigma_{n(2m+1)+2} = 2c + nr$$

remain less than $b+c$. Hence we can proceed from $\sigma_0=0$ to $\sigma_{n(2m+1)+1}=c+nr$ without passing any number greater than $2c+nr$.

A proof that the above condition is necessary as well as sufficient, or a counter-example, will be welcomed.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis. Mo. All manuscripts should be typewritten, wiih double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4021. *Proposed by Orrin Frink, Jr., Pennsylvania State College.*

The differential operator D^2+1 may be factored in many ways; for example, it may be written $(D+\cot x)(D-\cot x)$, or $(\sec x\cdot D)(\cos x\cdot D+\sin x)$, or $(\sin x\cdot D+2\cos x)(D\csc x)$. Show that the most general method of expressing the differential operator $(D+a)^2+b^2$ as the product of two real first order differential operators is given by the formula

$$(D+a)^2+b^2 = r^{-1}[D+a-b\tan(bx+c)-r'/r]\cdot r[D+a+b\tan(bx+c)],$$

where a, b, and c are real numbers, and r is a differentiable function of x.

4022. *Proposed by H. K. Humphrey, Winnetka, Ill.*

Derive a formula for the number of sets of p distinct integers taken from the first n positive integers, each set of p integers having the given sum S. Thus, for $n=16$, $p=4$, $S=34$, the number is 86.

4023. *Proposed by J. A. Greenwood, Duke University.*

Find an expression for the determinant of order $2n$

$$\begin{vmatrix} \theta I_n, & A_n \\ A_n, & \theta I_n \end{vmatrix},$$

where θI_n is a square matrix of order n having the variable θ in the principal diagonal and zeros elsewhere, and A_n is a square symmetric matrix of order n with a for each principal diagonal element, unity for the elements in the two parallels immediately above and below this principal diagonal, and zeros elsewhere.

4024. *Proposed by N. A. Court, University of Oklahoma.*

Given two twin tetrahedrons $(T)\equiv ABCD$, $(T')\equiv A'B'C'D'$ (see the proposer's *Modern Pure Solid Geometry*, p. 58, art. 191), consider the tetrahedron (A') formed by the face BCD of (T) and the three planes forming the trihedral angle A' of (T'); let (B'), (C'), (D') be the analogous tetrahedrons for the vertices B', C', D' of (T'). Show that the twelve-point spheres of the tetrahedrons (A'), (B'), (C'), (D') (*ibid.*, p. 251, art. 764) are tangent to the twelve-point sphere of (T).

4025. *Proposed by V. Thébault, San Sebastián, Spain.*

Let A_1', A_2', $\cdots$, A_{2n}' be the vertices of equilateral triangles constructed externally (or internally) on the sides A_1A_2, A_2A_3, $\cdots$, $A_{2n}A_1$ of a plane polygon of $2n$ sides $(P)\equiv A_1A_2\cdots A_{2n}$, and M_1, M_2, $\cdots$, M_n be the midpoints of the principal diagonals A_1A_{n+1}, A_2A_{n+2}, $\cdots$, A_nA_{2n} of (P). The midpoints M_1', M_2', $\cdots$, M_n' of the principal diagonals $A_1'A_{n+1}'$, $A_2'A_{n+2}'$, $\cdots$, $A_n'A_{2n}'$ of the polygon $(P')\equiv A_1'A_2'$, $\cdots$, A_{2n}' are the vertices of equilateral triangles constructed upon the sides of the polygon $(p)\equiv M_1M_2\cdots M_n$.

Generalize by replacing the equilateral triangles by similar isosceles triangles.

4026. *Proposed by V. Thébault, San Sebastián, Spain.*

(1) Construct a triangle ABC knowing a, A and given that the median and symmedian from A are perpendicular and parallel to two given directions. (2) Indicate the properties of this special triangle. (3) Let B' and C' be the projections of B and C on a variable straight line AP which cuts BC in P. The locus of the harmonic conjugate of P with respect to B' and C' is a right strophoid having the vertex A for a double point and tangent to the bisectors of angle A.

SOLUTIONS

3961 [1940, 323]. *Proposed by V. Thébault, San Sebastián, Spain.*

Each face angle of a given trihedral angle $O-XYZ$ is $\pi/3$, and on the respective edges the points A, B, C are located. Show that the Monge point of the tetrahedron $OABC$ describes a sphere as A, B, C vary on the edges so that $OA^2+OB^2+OC^2$ remains constant.

Solution by the Proposer.

This interesting problem is an application of a known formula, but seldom used, which expresses the circumradius R of a tetrahedron $OABC$ in terms of the lengths of the edges a', b', c' through O and the angles α, β, γ between these edges, namely,

$$4V^2R^2 = \sum a'^2 \sin^2\alpha + 2\sum b'c'(\cos\beta\cos\gamma - \cos\alpha), \tag{1}$$

where $V^2=1+2\cos\alpha\cos\beta\cos\gamma-(\cos^2\alpha+\cos^2\beta+\cos^2\gamma)$. Setting each of the

three angles equal to 60°, we have

$$8R^2 = 3\sum a'^2 - 2\sum b'c'. \tag{2}$$

After introducing the edges $BC=a$, $CA=b$, $AB=c$, we find that

$$8R^2 = 2\sum a^2 - \sum a'^2. \tag{3}$$

Then using the formulas (6) in this MONTHLY [1935, 430], we find that

$$8R^2 - \sum a'^2 = 2[\sum a^2 - \sum a'^2] = 8[R^2 - (O\Omega)^2],$$

where Ω is the Monge point. Finally we have

$$(O\Omega)^2 = \tfrac{1}{8}\sum a'^2,$$

and this justifies the proposition of the problem.

Editorial Note. A solution may be obtained without the use of the circumradius R; and for convenience we alter the notation. Let the three points be A_i, $(i=1, 2, 3)$, determined by the vectors $OA_i=a_i\mathbf{e}_i$, where $\mathbf{e}_i$ is the unit vector OE_i, the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ forming a right-hand system. Also, let $\mathbf{e}_k'$ be a unit vector orthogonal to $\mathbf{e}_i$ and $\mathbf{e}_j$ so that $\mathbf{e}_i$, $\mathbf{e}_j$, $\mathbf{e}_k'$ is a right-hand system. Since $OE_1E_2E_3$ is a regular tetrahedron, we have at once

$$\mathbf{e}_i \times \mathbf{e}_j = \frac{\sqrt{3}}{2}\mathbf{e}_k', \quad \mathbf{e}_i\cdot(\mathbf{e}_j \times \mathbf{e}_k) = \frac{\sqrt{2}}{2}, \quad \mathbf{e}_i'\cdot\mathbf{e}_j' = -\tfrac{1}{3}, \quad \mathbf{e}_i\cdot\mathbf{e}_i' = \sqrt{\tfrac{2}{3}}. \tag{1}$$

We now use the well known theorem that the plane through the midpoint of A_iA_j and perpendicular to the opposite edge OA_k passes through M, the Monge point of $OA_1A_2A_3$, with the vector $\mathbf{m}$. Then we have $2\mathbf{m}\cdot\mathbf{e}_k=a_i(\mathbf{e}_i\cdot\mathbf{e}_k)+a_j(\mathbf{e}_j\cdot\mathbf{e}_k)$, or

$$4\mathbf{m}\cdot\mathbf{e}_k = a_i + a_j. \tag{2}$$

The vector $\mathbf{m}$ may be written

$$\begin{aligned} &\sqrt{\tfrac{2}{3}}\mathbf{m} = \sum(\mathbf{e}_k\cdot\mathbf{m})\mathbf{e}_k', \qquad 4\sqrt{\tfrac{2}{3}}\mathbf{m} = \sum(a_i + a_j)\mathbf{e}_k', \\ &\tfrac{32}{3}\mathbf{m}^2 = \sum(a_i + a_j)^2 - \tfrac{2}{3}\sum(a_i + a_j)(a_j + a_k), \qquad \mathbf{m}^2 = \tfrac{1}{8}\sum a_i^2. \end{aligned} \tag{3}$$

Since in this problem the right side is constant, the Monge point M lies on a fixed sphere with center O.

The circumradius R of $OA_1A_2A_3$, where the angle between the edges $\mathbf{e}_i$, $\mathbf{e}_j$ is now α_k, may be found in a similar way. Since the circumcenter C, with the vector $\mathbf{c}$, lies on the plane perpendicular to OA_i at its midpoint, we have $2\mathbf{c}\cdot\mathbf{e}_i=a_i$, $(i=1, 2, 3)$. We now define the associated vectors by the equations $\mathbf{e}_i \times \mathbf{e}_j=\sin\alpha_k\mathbf{e}_k'$, $(\mathbf{e}_i \times \mathbf{e}_j)\cdot\mathbf{e}_k=V$, the volume of the parallelopiped with the three edges OE_1, OE_2, OE_3. Then we have

$$V\mathbf{c} = \sum (\mathbf{c}\cdot\mathbf{e}_i)\sin\alpha_i\mathbf{e}_i', \qquad 2V\mathbf{c} = \sum a_i \sin\alpha_i\mathbf{e}_i',$$

$$(4) \qquad 4V^2R^2 = \sum a_i^2\sin^2\alpha_i + 2\sum a_ja_k \sin\alpha_j \sin\alpha_k\, \mathbf{e}_j'\cdot\mathbf{e}_k',$$

$$= \sum a_i^2\sin^2\alpha_i + 2\sum a_ja_k(\cos\alpha_j\cos\alpha_k - \cos\alpha_i).$$

The determinant formula for V^2 has been given several times in this MONTHLY and here it reduces to

$$V^2 = 1 + 2\cos\alpha_1\cos\alpha_2\cos\alpha_3 - (\cos^2\alpha_1 + \cos^2\alpha_2 + \cos^2\alpha_3).$$

In the case of this problem where $\alpha_i = 60°$, we get

$$8R^2 = 3\sum a_i^2 - 2\sum a_ja_k.$$

3963 [1940, 399]. *Proposed by V. Thébault, San Sebastián, Spain.*

In an orthocentric tetrahedron the first sphere of twelve points is the locus of points the sum of whose powers, with respect to the spheres having as diameters the edges (or bimedians), is zero. Generalize.

Note. The bimedians are the straight line segments joining the midpoints of opposite edges.

Solution by Frank Ayres, Jr., Dickinson College.

In n-space, let $P_i \equiv (x_i^{(1)}, x_i^{(2)}, \cdots, x_i^{(n)}) \equiv (x_i)$, $(i = 1, 2, \cdots, n+1)$ be the vertices of an orthocentric $(n+1)$-point and $P \equiv (x)$ be its orthocenter. Let the radii of the $n+2$ mutually orthogonal hyperspheres having these points as centers be r_i and r, respectively. When the origin is taken at the center of the hypersphere circumscribing the $(n+1)$-point, it may be shown (Ayres, *On $n+2$ mutually orthogonal hyperspheres*, National Mathematics Magazine, vol. 10, 1936) that

$$\sum x_i^2 = R^2, \qquad \sum_{i\neq j} x_ix_j = R^2 - \tfrac{1}{2}(r_i^2 + r_j^2),$$

$$\sum x^2 = R^2 + nr^2, \qquad 2\sum x_ix = 2R^2 - r_i^2 + (n-1)r^2,$$

where $\sum$ indicates summation with respect to the superscripts and

$$R^2 = \frac{(n-1)^2}{4}r^2 + \tfrac{1}{4}\sum_i r_i^2$$

is the square of the radius of the circumscribing hypersphere.

The power of a point (X) with respect to the hypersphere whose diameter is the edge P_iP_j is given by

$$\sum\left(X - \frac{x_i + x_j}{2}\right)^2 - \sum\left(\frac{x_i - x_j}{2}\right)^2 = \sum X^2 - \sum X(x_i + x_j) + \sum x_ix_j;$$

hence, summing for all the edges, we have

$$(1)\qquad \binom{n+1}{2}(\sum X^2 + R^2) - n\sum X(x_1 + \cdots + x_{n+1}) - \frac{n}{2}\sum_i r_i^2.$$

The power of (X) with respect to the hypersphere whose diameter is the bimedian joining the midpoints of P_iP_j and P_sP_t is given by

$$\sum X^2 - \tfrac{1}{2}\sum X(x_i + x_j + x_s + x_t) + \tfrac{1}{4}\sum (x_i + x_j)(x_s + x_t);$$

hence, summing for all the bimedians, we have

$$(2)\qquad \binom{n+1}{4}(\sum X^2 + R^2) - \frac{1}{2}\binom{n}{3}\sum X(x_1 + \cdots + x_{n+1}) - \frac{1}{4}\binom{n}{3}\sum_i r_i^2.$$

Consider now the hypersphere G, whose center is the center of gravity of the orthocentric $(n+1)$-point, the square of whose radius is $\sum r_i^2/(n+1)^2$, and whose equation is

$$(3)\qquad \sum X^2 - \frac{2(n-1)}{n+1}\sum Xx + R^2 - \frac{1}{n+1}\sum_i r_i^2 = 0.$$

Since both (1) and (2) reduce to the left member of (3), we have the following:

The hypersphere G is the locus of points the sum of whose powers, with respect to the hyperspheres having the edges (or bimedians) of an orthocentric $(n+1)$-point as diameters, is zero.

A generalization may also be made as follows:

The first $n(n+1)$-point hypersphere of an orthocentric $(n+1)$-point is the locus of points the sum of whose powers, with respect to the hyperspheres centered at

$$\frac{n+1}{2(n-1)}\cdot\frac{x_i + x_j}{2} \text{ and with rad.}^2 = \frac{(n-3)^2}{16}\cdot r^2 + \frac{(n+1)^2}{32(n-1)}(r_i^2 + r_j^2),$$

is zero.

The first $n(n+1)$-point hypersphere is the locus of points the sum of whose powers, with respect to the hyperspheres centered at

$$\frac{n+1}{2(n-1)}\cdot\frac{x_i + x_j + x_s + x_t}{4} \text{ and with rad.}^2 = \frac{(n-3)^2}{16}r^2 + \frac{(n+1)^2(n-2)}{64(n-1)^2}(r_i^2 + r_j^2 + r_s^2 + r_t^2),$$

is zero.

In 3-space, these are the theorems of the problem.
Solved also by L. M. Kelly.

Editorial Note. Kelly considered only the tetrahedron, and his solution is based on two theorems, one by Thébault in his article, *On spheres associated with the tetrahedron,* in this MONTHLY, 1935, p. 433:

The sum of the powers of any point in space with respect to the spheres described on the edges of a tetrahedron as diameters is twice the sum of the powers of the same point with respect to the spheres described on the bimedians.

For the orthocentric tetrahedron, the three spheres with the bimedians as diameters coincide. The other theorem is due to Lagrange and states that, if masses m_i are at points A_i, the center of mass is denoted by G, the lengths A_iA_j are denoted by e_{ij}, and the distances of any point X from the points A_i are denoted by x_i, then

$$[XG]^2 = \frac{\sum m_i x_i^2}{\sum m_i} - \frac{\sum m_i m_j e_{ij}^2}{(\sum m_i)^2}, \qquad i \neq j. \tag{1}$$

For the present problem a unit mass is considered at each of the six midpoints of the edges of the orthocentric tetrahedron. His generalization is the above Thébault theorem.

The solution by Ayres can be formally simplified by using vectors $\mathbf{a}_i$ from the orthocenter H as origin to the vertices A_i of the simplex $A_1A_2, \cdots, A_{n+1}$, denoting by $\mathbf{g}$ and $\mathbf{c}$ the vectors of the centroid G and the circumcenter C of the simplex, and by R the circumradius. It is easily shown that $\mathbf{a}_i \cdot \mathbf{a}_j$ is a constant m, $i \neq j$. If we set $m = r^2$, the sphere with center H and radius r is sometimes called the polar sphere, or conjugate sphere, of the simplex. We shall set down some results which are easily deduced:

$$\begin{gathered} \mathbf{a}_i \cdot \mathbf{a}_j = m; \qquad 2\mathbf{c} = (n+1)\mathbf{g} = \sum \mathbf{a}_i; \qquad \mathbf{c}^2 = R^2 + nm; \\ \sum \mathbf{a}_i^2 = (n+1)^2 \mathbf{g}^2 - n(n+1)m = 4R^2 - n(n-3)m. \end{gathered} \tag{2}$$

The power of the point with the vector $\mathbf{x}$ with respect to the sphere on A_iA_j as diameter is $\mathbf{x}^2 - (\mathbf{a}_i + \mathbf{a}_j) \cdot \mathbf{x} + m$, and the sum of all such powers is

$$\binom{n+1}{2} [\mathbf{x}^2 - 2\mathbf{g} \cdot \mathbf{x} + m]. \tag{3}$$

The expression in the brackets set equal to zero is the equation of a sphere with center G, the square of whose radius is $\mathbf{g}^2 - m$; hence this sphere (G) is orthogonal to the polar sphere (H).

We now consider the special case where n is odd. Let $\mathbf{g}_{i_1}$ be the vector of the centroid of $(n+1)/2$ vertices A_{i_1}, and $\mathbf{g}_{i_2}$ that of the centroid of the remaining vertices A_{i_2}, and let b_{i_1} be the length of the segment joining the two centroids. Then $b_{i_1}^2 = (\mathbf{g}_{i_1} - \mathbf{g}_{i_2})^2$, $4\mathbf{g}^2 = (\mathbf{g}_{i_1} + \mathbf{g}_{i_2})^2$, and we then find that $b_{i_1}^2 = 4(\mathbf{g}^2 - m)$.

Hence these ${}_nC_{(n+1)/2}$ bimedians have equal lengths and bisect each other at G. They determine a single sphere (G) with center G which is the same as the one mentioned above. Other theorems may be obtained for the general simplex, replacing H by the Monge point M. The proof of the Lagrange theorem is quite easy, taking the origin of vectors at the point X.

The system of $n+2$ spheres used by Ayres contains the polar sphere (H) whose radius may be zero, real, or a pure imaginary number; the spheres (A_i) with center A_i are fixed by making each orthogonal to (H); hence we must have $r_i^2=\mathbf{a}_i^2-m$. Then (A_i) and (A_j) are orthogonal, since $r_i^2+r_j^2=\mathbf{a}_i^2+\mathbf{a}_j^2-2m=(\mathbf{a}_i-\mathbf{a}_j)^2$; and we now have a system of $n+2$ spheres orthogonal in pairs. Using (2) we easily find that $\sum r_i^2=4R^2-(n-1)^2m$, which is the same as an equation in the above solution.

The inverse of a point with vector $\mathbf{x}$ with respect to (H) has the vector $m\mathbf{x}/\mathbf{x}^2$. The sphere (C') which is the inverse of the circumsphere (C) is important and its equation can be easily obtained. The equations of the four spheres mentioned above and of their common radical plane $[P]$ are given below for reference:

$$(H):\ \mathbf{x}^2-m=0;\qquad (C'):\ n\mathbf{x}^2-2\mathbf{c}\cdot\mathbf{x}+m=0;$$

$$(G):\ \mathbf{x}^2-2\mathbf{g}\cdot\mathbf{x}+m=0;\qquad (C):\ \mathbf{x}^2-2\mathbf{c}\cdot\mathbf{x}+nm=0; \tag{4}$$

$$[P]\quad \mathbf{g}\cdot\mathbf{x}-m=0.$$

The isogonal conjugate of H is H' with vector $2\mathbf{c}/n$, and the projections of H, H' on the face opposite to A_i have the vectors $m\mathbf{a}_i/\mathbf{a}_i^2$, $[(n+1)\mathbf{g}-\mathbf{a}_i]/n$; these points are, respectively, the foot of the altitude from vertex A_i and the centroid of the face opposite to A_i. These last two points together with the point $\mathbf{a}_i/n$ lie on (C'), so that we have $3(n+1)$ of its points.

If H does not fall on a vertex, then m is not zero; if it does, say $H=A_{n+1}$, then $m=0$. In this latter case the equation of the face opposite to A_{n+1} is

$$\mathbf{x}\cdot\sum\mathbf{a}_i/\mathbf{a}_i^2=1;$$

whereas, if $m\neq 0$, the equation of the face opposite to A_i is $\mathbf{a}_i\cdot\mathbf{x}=m$, $(i=1, 2, \cdots, n+1)$; that is, it is the polar plane of A_i with respect to the polar sphere (H).

3964 [1940, 399]. *Proposed by V. Thébault, San Sebastián, Spain.*

The sum of the powers of the vertices of a tetrahedron, with respect to the Monge sphere of the circumscribed ellipsoid of Steiner, is equal to the negative of half the sum of the squares of the edges.

Editorial Note. If we take the centroid as origin of vectors $\mathbf{a}_i$ to the vertices of the tetrahedron $A_1A_2A_3A_4$ and denote by e_{ij} the length of the edge A_iA_j, then we have

$$0 = (\sum \mathbf{a}_i)^2 = \sum \mathbf{a}_i^2 + 2\sum \mathbf{a}_i \cdot \mathbf{a}_j,$$

$$\sum e_{ij}^2 = \sum (\mathbf{a}_i - \mathbf{a}_j)^2 = 3\sum \mathbf{a}_i^2 - 2\sum \mathbf{a}_i \cdot \mathbf{a}_j = 4\sum \mathbf{a}_i^2.$$

Let (D) be the sphere with center D at the end of vector $\mathbf{d}$, and radius r. In order for the sum of the powers of the vertices with respect to (D) to have the required value, we must have

$$\sum (\mathbf{a}_i^2 - 2\mathbf{d}\cdot\mathbf{a}_i + \mathbf{d}^2 - r^2) = \tfrac{1}{4}\sum e_{ij}^2 + 4(\mathbf{d}^2 - r^2) = -\tfrac{1}{2}\sum e_{ij}^2,$$

$$r^2 = \mathbf{d}^2 + \tfrac{3}{16}\sum e_{ij}^2.$$

The proposer stated that the sphere designated in the problem has its center at the centroid and the square of its radius is $(3/16)\sum e_{ij}^2$; hence, for this sphere $\mathbf{d}=0$.

3967 [1940, 491]. *Proposed by V. Thébault, San Sebastián, Spain.*

For a given triangle ABC a second triangle $A'B'C'$ is formed, where AA', BB', CC' are segments of altitudes and $AA'/BC = BB'/CA = CC'/AB = k$. (1) Show that the two triangles have the same centroid. (2) Examine the variation of the area of $A'B'C'$. (3) For what value of k do the two triangles have the same angle of Brocard? (4) If $k = \pm 1$, show that the centers of squares constructed exteriorly, or interiorly, on the sides of $A'B'C'$ are the vertices of ABC.

Solution by F. Underwood, University College, Nottingham, England.

The required results can be obtained by elementary coördinate geometry. Taking B as origin and the positive x-axis along BC, we find for the coördinates of the vertices of ABC and $A'B'C'$ the following:

$$\begin{aligned}
&A(c\cos B, \sin B); \quad B(0,0); \quad C(a,0);\\
&A'(c\cos B, c\sin B - ka); \quad B'(kb\sin C, kb\cos C); \quad C'(a - kc\sin B, kc\cos B);\\
&X_1 + X_2 + X_3 = c\cos B + a + k(b\sin C - c\sin B) = c\cos B + a;\\
&Y_1 + Y_2 + Y_3 = c\sin B - k(a - b\cos C - c\cos B) = c\sin B;
\end{aligned} \tag{1}$$

where X_1, Y_1 are coördinates of A', *etc.* The last two results prove that ABC and $A'B'C'$ have the same centroid.

Let Δ, Δ_1 denote the areas of ABC, $A'B'C'$; then, writing the determinant for $2\Delta_1$ and using reductions, we find that

$$2\Delta_1 = 2\Delta(1 + 3k^2) - kt, \quad t = a^2 + b^2 + c^2. \tag{2}$$

The algebraic minimum for Δ_1 is $\Delta - t^2/48\Delta$, when $k = t/12\Delta$. The area Δ_1 increases without limit as k takes on large positive or negative values.

Let the Brocard angles for the two triangles be ω, ω_1. Then $\cot\omega = \cot A + \cot B + \cot C = Rt/abc = t/4\Delta$. Denote the sides of $A'B'C'$ by a_1, b_1, c_1; then we have

$$a_1^2 = (B'C')^2 = (a - 2kb \sin C)^2 + k^2(c \cos B - b \cos C)^2$$
$$= (a - 4k\Delta/a)^2 + k^2(c^2 - b^2)^2/a^2.$$

Now $16\Delta^2+(c^2-b^2)^2=a^2(2b^2+2c^2-a^2)$; and hence

$$a_1^2 = a^2 - 8k\Delta + k^2(2b^2 + 2c^2 - a^2), \tag{3}$$

with similar expressions for b_1 and c_1. Then, setting $t_1=a_1+b_1+c_1$, we shall have

$$\cot \omega_1 = t_1/4\Delta = \frac{t(1 + 3k^2) - 24k\Delta}{4\Delta(1 + 3k^2) - 2kt}. \tag{4}$$

If $\omega=\omega_1$, we find that after reductions, $kt^2=48k\Delta^2$, and there are two cases; either $k=0$, or $t^2=48\Delta$. The second case requires that $(a^2+b^2+c^2)^2=3(2\sum b^2c^2 - \sum a^4)$, or $\sum a^4-\sum b^2c^2=0$, or

$$(b^2 - c^2)^2 + (c^2 - a^2)^2 + (a^2 - b^2)^2 = 0.$$

Hence ABC and $A'B'C'$ have the same Brocard angle if and only if the two triangles coincide or ABC is equilateral.

When $k=\pm 1$, the equation (3) gives

$$a_1^2 = \mp 8\Delta + (2b^2 + 2c^2) = 2(b^2 + c^2 \mp 2bc \sin A).$$

For $k=1$, we have

$$(AB')^2 = (c \cos B - b \sin C)^2 + (c \sin B - b \cos C)^2 = b^2 + c^2 - 2bc \sin A;$$
$$(AC')^2 = (a - c \sin B - c \cos B)^2 + (c \sin B - c \cos B)^2$$
$$= (b \cos C - c \sin B)^2 + (b \sin C - c \cos B)^2 = b^2 + c^2 - 2bc \sin A.$$

Hence A is the center of one of the squares with side $B'C'$. For $k=-1$, similar computations give for a_1^2, $(AB')^2$, $(AC')^2$ the above results with the minus sign changed to plus. Hence, in this case also, A is the center of one of the squares with side $B'C'$. This completes the proof of (4).

Editorial Note. We may also write

$$\Delta_1 = \Delta(3k^2 - 2k \cot \omega + 1),$$

so that $\Delta_1=0$ for $3k=\cot \omega \pm \sqrt{\cot^2\omega-3}$. Hence, if $\cot \omega=2$ and $k=1/3$ or 1, we have $\Delta_1=0$.

The proposer gave the above expressions for $\cot \omega$, and considered k as positive when AA' is directed toward the opposite side, *etc.* He said that part (1) follows easily on observing that the points A', B', C' are the centers of similar rectangles constructed interiorly (or exteriorly) on the sides of $A_2B_2C_2$, the anticomplementary triangle of ABC. (2) The area of $A'B'C'$ vanishes under certain conditions: thus for $k=1$ and $\cot \omega=2$, the points A', B', C' are collinear.

(3) It is necessary and sufficient that $\cot \omega = 3$. (4) The points A', B', C' are centers of squares constructed interiorly (or exteriorly) on B_2C_2, C_2A_2, A_2B_2, and triangle $A'B'C'$ is the anticomplement of the one whose vertices are centers of squares on BC, CA, AB. The desired results follow from these facts.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to C. O. Oakley, Haverford College, Haverford, Pennsylvania.

A few more copies of the MONTHLY for October 1940 are needed to replace losses to members and libraries in the British Isles. Copies for this purpose sent to the Secretary of the Association will be forwarded by him.

The editors of the Annals of Mathematics have taken an action similar to that of the Duke University Press and have agreed to continue to allow the half rate after 1942 to those whose membership in the Association and whose subscription to the Annals of Mathematics are unbroken from 1942 to the year in question. This arrangement is expected to be permanent. Those members of the Association who contemplate subscribing for either of these journals should, therefore, do so before the end of this year at the rate of $2.50 a year for the Annals of Mathematics and $2.00 a year for the Duke Mathematical Journal.

Professor G. D. Birkhoff of Harvard University has been elected an honorary member of the London Mathematical Society.

The honorary degree of doctor of laws has been conferred by Lehigh University upon Dean R. G. D. Richardson of Brown University.

Professor L. C. Bagby of Linsly Institute of Technology has been appointed educational director of the air school of the Hartung Aircraft Corporation, Detroit, with a special assignment to mathematics, drafting and course planning.

Dr. H. R. Branson of Dillard University has been appointed an assistant professor at Howard University.

Dr. W. B. Caton of Athens College, Alabama, has been appointed acting head of the department at Southwestern College, Winfield, Kansas.

Associate Professor J. E. Davis of Drexel Institute of Technology has been promoted to a professorship.

Dr. F. G. Dressel of Duke University has been promoted to an assistant professorship.

Professor Emeritus W. F. Durand of Stanford University has been appointed a member of the National Advisory Committee on Aeronautics.

Dr. A. S. Galbraith of the University of Rochester has been appointed an assistant professor at Colby College.

Professor I. M. Hostetter of Howard College, Alabama, has been appointed an assistant professor at Oregon State College.

Dr. L. F. Ollmann of the College of Wooster has been appointed an associate professor and head of the department at Hofstra College.

Dr. W. J. Schart of Ohio State University has been appointed professor of mathematics at the Aviation Cadet Replacement Center, Maxwell Field, Alabama.

Dr. C. E. Sealander of the State University of Iowa has been appointed an assistant professor at the University of South Dakota.

J. W. Sheedy of Michigan State College has been promoted to an assistant professorship.

Associate Professor E. C. Stopher of Ashland College has accepted a position at Brockport, New York, State Normal School.

Dr. W. F. Whitmore of the University of California has accepted a position at the Naval Ordnance Laboratory, Washington, D. C.

Assistant Professor Louise A. Wolf of the University of Wisconsin Extension Division has been appointed lecturer at the University of Wisconsin.

The following appointments to instructorships have been announced:
Colorado College: Dr. Margaret M. Hansman
Cornell University: Dr. Joseph Lehner, J. C. Smith
Frostburg, Maryland, State Teachers College: Dr. W. N. Hallett
Illinois Institute of Technology: Dr. W. S. Snyder
Indiana Technical College: H. S. Kieval
University of Maryland: Dr. L. C. Hutchinson
Mundelein College, Chicago: E. L. Godfrey
New Mexico College of A. and M. A.: J. A. Joseph
University of Oklahoma: Dr. Ingo Maddaus
Oregon State College: Dr. Rhoda Manning
Pennsylvania State College: Dr. Abraham Schwartz
University of Pennsylvania: E. K. Ritter
Purdue University: Dr. J. W. T. S. Youngs, Dr. G. W. Whitehead
United States Naval Academy: T. J. Benac, Dr. E. E. Betz
Wells College: Dr. Mary D. Clement
University of Wisconsin: H. N. Laden
College of Wooster: H. L. Meyer, Jr.
Yale University: Dr. R. P. Dilworth

Professor R. L. Charles of Franklin and Marshall College died December 13, 1941, at the age of fifty-six. He was a charter member of the Mathematical Association.

Dr. W. P. Durfee, who was head of the department of mathematics at Hobart College from 1884 until his retirement in 1929, and dean of the college from 1888 to 1925, died December 17, 1941, in his eighty-seventh year. He was a charter member of the Association.

Professor C. E. Magnusson of the University of Washington died July 10, 1941.

E. J. Maurus, professor of mathematics at the University of Notre Dame from 1897 to 1939, died November 26, 1941, at the age of sixty-nine. He had been a member of the Mathematical Association for twenty-one years.

Dr. U. G. Mitchell, professor of mathematics and until last September chairman of the department at the University of Kansas, died early January 1, 1942, at the age of sixty-nine years. He was a charter member of the Association and served as an associate editor of the MONTHLY from 1915 through 1921.

MEETINGS OF THE ASSOCIATION AND ITS SECTIONS

Twenty-seventh Annual Meeting, New York, N. Y., December 30–31, 1942.

The following is a list of the Sections of the Association, with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS, Decatur, May 8–9, 1942
INDIANA, Crawfordsville, May 1–2, 1942
IOWA, Mt. Pleasant, April 17–18, 1942
KANSAS, Hays, March 27–28, 1942
KENTUCKY, Lexington, April 11, 1942
LOUISIANA-MISSISSIPPI, Jackson, Miss., March 6–7, 1942
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Ashland, Va., May 1942
METROPOLITAN NEW YORK, New York, April 18, 1942
MICHIGAN
MINNESOTA
MISSOURI, Kansas City, April 17, 1942
NEBRASKA, Omaha, May 9, 1942
NORTHERN CALIFORNIA, San Francisco, Jan 30, 1943
OHIO, Columbus, April 2, 1942
OKLAHOMA, Oklahoma City, Feb. 13, 1942
PHILADELPHIA, Philadelphia, Nov. 28, 1942
ROCKY MOUNTAIN, Golden Colo., April 17–18, 1942
SOUTHEASTERN, Emory University, Ga., March 26–27, 1942
SOUTHERN CALIFORNIA, Los Angeles, March 14, 1942
SOUTHWESTERN, State College, N. M., April 27–28, 1942
TEXAS, Lubbock, April 3–4, 1942
UPPER NEW YORK STATE, Rochester, May 2, 1942
WISCONSIN, Oshkosh, May 2, 1942

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